

Transfer matrix approach to 1d random band matrices: density of states

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Abstract

We study the special case of $n \times n$ 1D Gaussian Hermitian random band matrices, when the covariance of the elements is determined by the matrix $J = (-W^2\Delta + 1)^{-1}$. Assuming that $n \geq CW \log W \gg 1$, we prove that the averaged density of states coincides with the Wigner semicircle law up to the correction of order W^{-1} .

1 Introduction

We consider Hermitian $n \times n$ matrices H_n whose entries H_{ij} are random complex Gaussian variables with mean zero such that

$$\mathbf{E}\{H_{ij}H_{lk}\} = \delta_{ik}\delta_{jl}J_{ij}, \quad (1.1)$$

where

$$J_{ij} = (-W^2\Delta + 1)^{-1}_{ij}, \quad (1.2)$$

and $\mathbf{E}\{\dots\}$ denotes the average with respect to the probability distribution of H_n . Here Δ is the discrete Laplacian on $\mathcal{L} = [1, n] \cap \mathbb{Z}$ with Neumann boundary conditions:

$$(-\Delta f)_j = \begin{cases} f_1 - f_2, & j = 1; \\ 2f_j - f_{j-1} - f_{j+1}, & j = 2, \dots, n-1; \\ f_n - f_{n-1}, & j = n. \end{cases}$$

The probability law of H_n can be written in the form

$$P_n(dH_n) = \prod_{1 \leq i < j \leq n} \frac{dH_{ij}d\bar{H}_{ij}}{2\pi J_{ij}} e^{-\frac{|H_{ij}|^2}{J_{ij}}} \prod_{i=1}^n \frac{dH_{ii}}{\sqrt{2\pi J_{ii}}} e^{-\frac{H_{ii}^2}{2J_{ii}}}. \quad (1.3)$$

It is easy to see that $J_{ij} \approx C_1 W^{-1} \exp\{-C_2|i-j|/W\}$, so it is exponentially small when $|i-j| \gg W$. Thus matrices H_n can be considered as a special case of random band matrices with the band width W . The same model can be defined similarly in any dimension d (then $i, j \in \mathcal{L} = [1, n]^d \cap \mathbb{Z}^d$).

Let $\lambda_1^{(n)}, \dots, \lambda_n^{(n)}$ be the eigenvalues of H_n . Define their Normalized Counting Measure (NCM) as

$$\mathcal{N}_n(I) = \frac{1}{n} \#\{\lambda_j^{(n)} \in I, j = 1, \dots, n\}, \quad \mathcal{N}_n(\mathbb{R}) = 1,$$

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where I is an arbitrary interval of the real axis. It was shown in [2, 14] that for 1d RBM (even for more general than (1.2) form of the variance) \mathcal{N}_n converges in probability, as $n, W \rightarrow \infty$, to a non-random measure \mathcal{N} , which is absolutely continuous, and its density ρ is given by the well-known Wigner semicircle law (the same result is valid for Wigner ensembles, in particular, for Gaussian ensembles GUE, GOE):

$$\rho_{sc}(E) = \begin{cases} (2\pi)^{-1} \sqrt{4 - E^2}, & E \in [-2, 2]; \\ 0, & |E| \geq 2. \end{cases} \quad (1.4)$$

A substantial interest to random band matrices is caused by the fact that they are natural intermediate models between random Schrödinger matrices $H_{RS} = -\Delta + \lambda V$, in which the randomness only appears in the diagonal potential V (λ is a small parameter which measures the strength of the disorder) and mean-field random matrices such as $n \times n$ Wigner matrices, i.e. Hermitian random matrices with i.i.d elements. In particular, RBM can be used to model the Anderson metal-insulator phase transition. Moreover, it is conjectured (see [5, 13]) that the transition for RBM can be investigated even in $d = 1$ by varying the band width W . It is expected that 1d RBM changes the spectral local behaviour of random operator type with Poisson local eigenvalue statistics corresponding to localized eigenstates (for $W \ll \sqrt{n}$) to the local spectral behaviour of the Gaussian Unitary Matrix type corresponding to delocalized eigenstates (for $W \gg \sqrt{n}$) (for more details on these conjectures see e.g. [21]). Some partial results about localization and delocalization (in a weak sense) for general RBM was obtained in [16], [10], [11]. Universality of the gap distribution for $W \sim n$ was also obtained in a recent paper [3]. However, the question of the existing of a crossover in RBM is still open even for $d = 1$.

One of the approaches, which allows to work with random operators with non-trivial spatial structures, is supersymmetry techniques (SUSY) based on the representation of the determinant as an integral over the Grassmann variables. This method is widely used in the physics literature and is potentially very powerful, but the rigorous control of the integral representations, which can be obtained by this method, is quite difficult. However, it can be done rigorously for some special class of RBM. For instance, by using SUSY the detailed information about the averaged density of states of ensemble (1.1) – (1.3) in dimension 3 including local semicircle law at arbitrary short scales and smoothness in energy (in the limit of infinite volume and fixed large band width W) was obtained in [7]. Moreover, by applying SUSY approach in [18], [17] the crossover in this model (in 1d) was proved for the correlation functions of characteristic polynomials. In addition, the rigorous application of SUSY to the Gaussian RBM which has the special block-band structure (special case of Wegner’s orbital model) was developed in [19], where the universality of the bulk local regime for $W \sim n$ was proved. Combining this approach with Green’s function comparison strategy the delocalization (in a strong sense) for $W \gg n^{6/7}$ has been proved in [1] for the block band matrices with rather general element’s distribution.

In this paper we develop the method of [17], which combines the SUSY techniques with a transfer matrix approach (see also [8]). The final goal is to extend this method from the correlation function of characteristic polynomials to usual correlation functions of (1.1) – (1.3), which allows to study the crossover of local eigenvalue statistics for 1d RBM. To this end we have to study the transfer operator involving not only the complex, but also the Grassmann variables (for the second correlation functions it involves 8 Grassmann variables). At the present paper we make the first step in this direction: we study the transfer operator appearing from the integral representation of the density of states of ensemble (1.1) – (1.3)

in 1d (see (2.11) below), which has only two Grassmann variables. The supersymmetric transfer matrix formalism was first suggested by Efetov (see [9]), and it was successfully applied rigorously to the density of states of some models (see e.g. [4], [6]).

According to the property of the Stieltjes transform, the averaged density of states is given by

$$\bar{\rho}_n(E) = \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \mathbf{E} \left\{ n^{-1} \text{Im Tr} (E_\varepsilon - H_n)^{-1} \right\}, \quad (1.5)$$

where $E_\varepsilon = E - i\varepsilon$, $E \in (-2, 2)$.

Thus, we are interested in

$$\bar{g}_n(E) = \lim_{\varepsilon \downarrow 0} \bar{g}_n(E_\varepsilon) \quad (1.6)$$

with

$$\bar{g}_n(E_\varepsilon) = \mathbf{E} \left\{ n^{-1} \text{Tr} (E_\varepsilon - H_n)^{-1} \right\} = \mathbf{E} \left\{ - \frac{\partial}{\partial x} \frac{\det(E_\varepsilon - H_n)}{\det(E_\varepsilon + x/n - H_n)} \Big|_{x=0} \right\}.$$

Our main result is

Theorem 1.1 *Let H_n be 1d Gaussian RBM defined in (1.1) – (1.3) with $n \geq C_0 W \log W$, and let $|E| \leq 4\sqrt{2}/3 \approx 1.88$. Then for $\bar{g}_n(E)$ defined in (1.6) we have*

$$|\bar{g}_n(E) - g_{sc}(E)| \leq C/W,$$

where

$$g_{sc}(E) = \lim_{\varepsilon \downarrow 0} \int \frac{\rho_{sc}(\lambda) d\lambda}{E - i\varepsilon - \lambda} = \frac{E + i\sqrt{4 - E^2}}{2}.$$

In particular,

$$|\bar{\rho}_n(E) - \rho_{sc}(E)| \leq C/W,$$

where $\bar{\rho}_n(E)$ is an averaged density of states (1.5), and ρ_{sc} is defined in (1.4).

Note that Theorem 1.1 gives

$$|\bar{g}_n(E - i\varepsilon) - g_{sc}(E - i\varepsilon)| \leq C/W \quad (1.7)$$

uniformly in any arbitrary small $\varepsilon \geq 0$. As it was mentioned above, similar asymptotics (with correction C/W^2) for RBM of (1.1) in 3d was obtained in [7], however their method cannot be applied to 1d case. All other previous results about the density of states for RBM deal with $\varepsilon \gg W^{-1}$ or bigger (for fixed $\varepsilon > 0$ the asymptotics (1.7) follows from the results of [2]; [10] gives (1.7) with $\varepsilon \gg W^{-1/3}$; [20] yields (1.7) for 1d RBM with Bernoulli elements distribution for $\varepsilon \geq W^{-0.99}$, and [12] proves similar to (1.7) asymptotics with correction $1/(W\varepsilon)^{1/2}$ for $\varepsilon \gg 1/W$). On the other hand, the methods of [10], [12] allow to control $n^{-1} \text{Tr} (E_\varepsilon - H_n)^{-1}$ and $(E_\varepsilon - H_n)_{xy}^{-1}$ for $\varepsilon \gg W^{-1}$ without expectation, which gives some information about the localization length. This cannot be obtained from Theorem 1.1, since it requires estimates on $\mathbb{E}\{|(E_\varepsilon - H_n)_{xy}^{-1}|^2\}$. Similar estimates for $\varepsilon \approx n^{-1}$ is required to work with the second correlation function, and we hope it will be the aim of the next paper.

The paper is organized as follows. In Section 2 we re-derive an integral representation for $\bar{g}_n(E)$ obtained in [7]. In Section 3 we rewrite this representation in terms of the transfer operator \mathcal{K} (see (3.11)). Section 4 deals with the analysis of the operator \mathcal{K} (see Theorem 4.2) and the proof of Theorem 1.1. In Section 5 we prove an important preliminary result needed for Section 4.

2 Integral representation

In this section we obtain an integral representation for $\bar{g}_n(E)$ of (1.6) by using integration over the Grassmann variables. Such representation for the density of states of ensemble (1.1) – (1.2) was obtained in [7] in any dimension d . For the reader convenience we repeat here the derivation of the integral representation for $d = 1$.

Integration over the Grassmann variables has been introduced by Berezin and is widely used in the physics literature (see e.g. [9]). A brief outline of the techniques can be found e.g. in [9].

Let A be an ordinary matrix with a positive Hermitian part. The following Gaussian integral is well-known:

$$\int \exp \left\{ - \sum_{j,k=1}^n A_{j,k} z_j \bar{z}_k \right\} \prod_{j=1}^n \frac{d\Re z_j d\Im z_j}{\pi} = \frac{1}{\det A}. \quad (2.1)$$

One of the most important formulas of the Grassmann variables theory is the analog of (2.1) for the Grassmann variables (see [9]):

$$\int \exp \left\{ - \sum_{j,k=1}^n A_{j,k} \bar{\psi}_j \psi_k \right\} \prod_{j=1}^n d\bar{\psi}_j d\psi_j = \det A, \quad (2.2)$$

where A now is any $n \times n$ matrix. Combining these two formulas one can obtain also

$$\int \exp \left\{ - \Phi^+ F \Phi \right\} \prod_{j=1}^n \frac{d\Re z_j d\Im z_j d\bar{\psi}_j d\psi_j}{\pi} = \text{sdet } F, \quad (2.3)$$

where $\Phi = (z_1, \dots, z_n, \psi_1, \dots, \psi_n)^t$,

$$F = \begin{pmatrix} B & \Sigma^+ \\ \Sigma & A \end{pmatrix}, \quad \text{sdet } F = \frac{\det(A - \Sigma B^{-1} \Sigma^+)}{\det B},$$

and $B > 0$, A are $n \times n$ complex matrices, Σ, Σ^+ are $n \times n$ matrices of anticommuting elements of Grassmann algebra.

Using (2.1) – (2.2), we can rewrite

$$\begin{aligned} \mathbf{E} \left\{ \frac{\det(E_\varepsilon - H_n)}{\det(E_\varepsilon + x/n - H_n)} \right\} &= \mathbf{E} \left\{ \int e^{i \sum_{j,k=1}^n (E_\varepsilon - H_n)_{jk} \bar{\psi}_j \psi_k + i \sum_{j,k=1}^n (E_\varepsilon + x/n - H_n)_{jk} \bar{\phi}_j \phi_k} d\Phi \right\} \\ &= \int \exp \left\{ \sum_j \left(i E_\varepsilon \bar{\psi}_j \psi_j + i (E_\varepsilon + x/n) \bar{\phi}_j \phi_j \right) \right\} \\ &\times \mathbf{E} \left\{ \exp \left\{ - \sum_{j < k} \left(i \Re H_{jk} \cdot (\eta_{jk} + \eta_{kj}) - \Im H_{jk} \cdot (\eta_{jk} - \eta_{kj}) \right) - i \sum_j H_{jj} \cdot \eta_{jj} \right\} \right\} d\Phi, \end{aligned}$$

where $\{\psi_j\}_{j=1}^n$ are Grassmann (i.e. anticommuting) variables, $\{\phi_j\}_{j=1}^n \in \mathbb{C}^n$,

$$\begin{aligned} \eta_{jk} &= \bar{\psi}_j \psi_k + \bar{\phi}_j \phi_k, \\ d\Phi &= \prod_{q=1}^n \frac{d\Re \phi_q d\Im \phi_q}{\pi} \prod_{q=1}^n d\bar{\psi}_q d\psi_q. \end{aligned}$$

Taking the average according to (1.3), we get

$$\begin{aligned}
\mathbf{E} \left\{ \frac{\det(E_\varepsilon - H_n)}{\det(E_\varepsilon + x/n - H_n)} \right\} &= \int e^{\sum_j (iE_\varepsilon \bar{\psi}_j \psi_j + i(E_\varepsilon + x/n) \bar{\phi}_j \phi_j) - \frac{1}{2} \sum_{j,k} J_{jk} \eta_{jk} \eta_{kj}} d\Phi \\
&= \int d\Phi \exp \left\{ \sum_j (iE_\varepsilon \bar{\psi}_j \psi_j + i(E_\varepsilon + x/n) \bar{\phi}_j \phi_j) \right\} \\
&\exp \left\{ \frac{1}{2} \sum_{j,k} J_{jk} \bar{\psi}_j \psi_j \cdot \bar{\psi}_k \psi_k - \sum_{j,k} J_{jk} \bar{\psi}_j \phi_j \cdot \psi_k \bar{\phi}_k - \frac{1}{2} \sum_{j,k} J_{jk} \bar{\phi}_j \phi_j \cdot \bar{\phi}_k \phi_k \right\}.
\end{aligned} \tag{2.4}$$

To convert the quartic interaction in (2.4) into a quadratic one we perform a standard Hubbard-Stratonovich transformation:

$$\begin{aligned}
\exp \left\{ \frac{1}{2} \sum_{j,k} J_{jk} \bar{\psi}_j \psi_j \cdot \bar{\psi}_k \psi_k \right\} &= \frac{\det^{-1/2} J}{(2\pi)^{n/2}} \int \exp \left\{ -\frac{1}{2} \sum_{j,k} J_{jk}^{-1} a_j a_k + \sum_j a_j \bar{\psi}_j \psi_j \right\} \prod_{j=1}^n da_j; \\
\exp \left\{ -\frac{1}{2} \sum_{j,k} J_{jk} \bar{\phi}_j \phi_j \cdot \bar{\phi}_k \phi_k \right\} &= \frac{\det^{-1/2} J}{(2\pi)^{n/2}} \int \exp \left\{ -\frac{1}{2} \sum_{j,k} J_{jk}^{-1} b_j b_k - i \sum_j b_j \bar{\phi}_j \phi_j \right\} \prod_{j=1}^n db_j; \\
\exp \left\{ -\sum_{j,k} J_{jk} \bar{\psi}_j \phi_j \cdot \psi_k \bar{\phi}_k \right\} \\
&= \det J \int \exp \left\{ -\sum_{j,k} J_{jk}^{-1} \bar{\rho}_j \rho_k - i \sum_j \bar{\rho}_j \psi_j \bar{\phi}_j + i \sum_j \rho_j \bar{\psi}_j \phi_j \right\} \prod_{j=1}^n d\bar{\rho}_j d\rho_j.
\end{aligned}$$

This gives

$$\begin{aligned}
\mathbf{E} \left\{ \frac{\det(E_\varepsilon - H_n)}{\det(E_\varepsilon + x/n - H_n)} \right\} &= \frac{1}{(2\pi)^n} \int d\Phi d\bar{X} e^{\sum_j (iE_\varepsilon \bar{\psi}_j \psi_j + i(E_\varepsilon + x/n) \bar{\phi}_j \phi_j)} \\
&\exp \left\{ -\frac{1}{2} \sum_{j,k} J_{jk}^{-1} \text{Str } X_j X_k + \sum_j a_j \bar{\psi}_j \psi_j - i \sum_j b_j \bar{\phi}_j \phi_j - i \sum_j \bar{\rho}_j \psi_j \bar{\phi}_j + i \sum_j \rho_j \bar{\psi}_j \phi_j \right\},
\end{aligned} \tag{2.5}$$

where

$$X_j = \begin{pmatrix} b_j & \bar{\rho}_j \\ \rho_j & ia_j \end{pmatrix}, \quad \text{Str} \begin{pmatrix} x & \bar{\sigma} \\ \sigma & y \end{pmatrix} = x - y, \tag{2.6}$$

and

$$d\bar{X} = \prod_{j=1}^n dX_j, \quad dX_j = da_j db_j d\bar{\rho}_j d\rho_j.$$

Here a_j, b_j are complex variables, and $\bar{\rho}_j, \rho_j$ are Grassmann variables. Applying (2.3) to integrate over $d\Phi$ in (2.5), we obtain

$$\begin{aligned}
\mathbf{E} \left\{ \frac{\det(E_\varepsilon - H_n)}{\det(E_\varepsilon + x/n - H_n)} \right\} \\
= \frac{1}{(2\pi)^n} \int \exp \left\{ -\frac{1}{2} \sum_{j,k} J_{jk}^{-1} \text{Str } X_j X_k \right\} \prod_{j=1}^n \text{sdet}(X_j - \Lambda_x) d\bar{X}, \tag{2.7}
\end{aligned}$$

where

$$\Lambda_x = \begin{pmatrix} E_\varepsilon + x/n & 0 \\ 0 & E_\varepsilon \end{pmatrix}.$$

Substituting (1.2) and (2.6), we can rewrite (2.7) as

$$\begin{aligned} \lim_{\varepsilon \downarrow 0} \mathbf{E} \left\{ \frac{\det(E_\varepsilon - H_n)}{\det(E_\varepsilon + x/n - H_n)} \right\} &= \frac{1}{(2\pi)^n} \int \exp \left\{ -\frac{W^2}{2} \sum_{j=1}^n \left((a_j - a_{j-1})^2 + (b_j - b_{j-1})^2 \right) \right\} \\ &\exp \left\{ -W^2 \sum_{j=1}^n (\bar{\rho}_j - \bar{\rho}_{j-1})(\rho_j - \rho_{j-1}) - \frac{1}{2} \sum_{j=1}^n (a_j^2 + b_j^2) - \sum_{j=1}^n \bar{\rho}_j \rho_j \right\} \\ &\prod_{j=1}^n \frac{ia_j - E_\varepsilon}{b_j - E_\varepsilon - x/n} \cdot \prod_{j=1}^n \left(1 + \frac{\bar{\rho}_j \rho_j}{(ia_j - E_\varepsilon)(b_j - E_\varepsilon - x/n)} \right) d\bar{X} \\ &= \frac{1}{(2\pi)^n} \int d\bar{X} \exp \left\{ -\frac{W^2}{2} \sum_{j=1}^n \left((a_j - a_{j-1})^2 + (b_j - b_{j-1})^2 \right) \right\} \\ &\times \exp \left\{ -\frac{1}{n} \sum_{j=1}^n \left((b_j + i\sqrt{4 - E^2}/2)x + x^2/2n \right) - \sum_{j=1}^n (f_a(a_j) + f_b(b_j)) \right\} \\ &\times \exp \left\{ -W^2 \sum_{j=1}^n (\bar{\rho}_j - \bar{\rho}_{j-1})(\rho_j - \rho_{j-1}) - \sum_{j=1}^n \bar{\rho}_j \rho_j L(a_j, b_j) \right\}, \end{aligned} \quad (2.8)$$

where, to obtain the last equality, we use

$$1 + \frac{\bar{\rho}_j \rho_j}{(ia_j - E_\varepsilon)(b_j - E_\varepsilon - x/n)} = \exp \left\{ \frac{\bar{\rho}_j \rho_j}{(ia_j - E_\varepsilon)(b_j - E_\varepsilon - x/n)} \right\},$$

change $b_j \rightarrow b_j + x/n + \frac{i\sqrt{4 - E^2}}{2}$, $a_j \rightarrow a_j - iE_\varepsilon/2$ and put $\varepsilon = 0$. In the last line of (2.8) we put

$$f_a(x) = (x - iE/2)^2/2 - \log(ix - E/2) + C^*; \quad (2.9)$$

$$f_b(x) = (x + i\sqrt{4 - E^2}/2)^2/2 + \log(x - E + i\sqrt{4 - E^2}/2) - C^*;$$

$$C^* = (E/2 + i\sqrt{4 - E^2}/2)^2/2 + \log(-E/2 + i\sqrt{4 - E^2}/2);$$

$$L(x, y) = 1 - \frac{1}{(ix - E/2)(y - E + i\sqrt{4 - E^2}/2)}. \quad (2.10)$$

Taking the derivative of (2.8) with respect to x and putting $x = 0$, we get

$$\begin{aligned} \bar{g}_n(E) &= \frac{1}{(2\pi)^n} \int \left(\sum_{j=1}^n \frac{b_j + i\sqrt{4 - E^2}/2}{n} \right) \cdot \exp \left\{ -\frac{W^2}{2} \sum_{j=1}^n \left((a_j - a_{j-1})^2 + (b_j - b_{j-1})^2 \right) \right\} \\ &\exp \left\{ -W^2 \sum_{j=1}^n (\bar{\rho}_j - \bar{\rho}_{j-1})(\rho_j - \rho_{j-1}) - \sum_{j=1}^n (f_a(a_j) + f_b(b_j)) - \sum_{j=1}^n \bar{\rho}_j \rho_j L(a_j, b_j) \right\} d\bar{X}. \end{aligned} \quad (2.11)$$

Besides, putting $x = 0$ (2.8) we have

$$1 = \frac{1}{(2\pi)^n} \int \exp \left\{ -\frac{W^2}{2} \sum_{j=1}^n \left((a_j - a_{j-1})^2 + (b_j - b_{j-1})^2 \right) \right\} \\ \exp \left\{ -W^2 \sum_{j=1}^n (\bar{\rho}_j - \bar{\rho}_{j-1})(\rho_j - \rho_{j-1}) - \sum_{j=1}^n (f_a(a_j) + f_b(b_j)) - \sum_{j=1}^n \bar{\rho}_j \rho_j L(a_j, b_j) \right\} d\bar{X}. \quad (2.12)$$

Now let us study the stationary points of f_a, f_b :

Lemma 2.1 (i) *The function $\Re f_a(x)$, $x \in \mathbb{R}$ attains its minimum at*

$$a_{\pm} = \pm \frac{\sqrt{4 - E^2}}{2}. \quad (2.13)$$

Moreover,

$$f_a(a_+) = f'_a(a_{\pm}) = 0, \quad f_a(a_-) = \frac{iE\sqrt{4 - E^2}}{2} + 2\log(-E/2 + i\sqrt{4 - E^2}/2) \in i\mathbb{R}.$$

(ii) *For $|E| < 4\sqrt{2}/3 \approx 1.88$, the function $\Re f_b(x)$, $x \in \mathbb{R}$ attains its minimum at*

$$b_s = \frac{E}{2}. \quad (2.14)$$

Moreover,

$$f_b(b_s) = f'_b(b_s) = 0.$$

The proof of the lemma is straightforward and it is omitted here.

Define also

$$c_{\pm} := f''_a(a_{\pm})/2, \quad (2.15)$$

and note that $\Re c_{\pm} > 0$, $\arg c_{\pm} \in (-\pi/2, \pi/2)$. Besides,

$$f''_b(b_s) = 2c_+; \quad (2.16)$$

$$L^+ := L(a_+, b_s) = 1 - \left(\frac{E - i\sqrt{4 - E^2}}{2} \right)^2 = 2c_+;$$

$$L^- := L(a_-, b_s) = 0.$$

3 Transfer matrix approach

Expanding the exponent into the series it is easy to see that

$$\int \exp \left\{ -W^2(\bar{\rho}' - \bar{\rho})(\rho' - \rho) - L\bar{\rho}'\rho' \right\} (q_1 + q_2\bar{\rho}'\rho' + q_3\bar{\rho}' + q_4\rho') d\rho' d\bar{\rho}' \\ = W^2 \left(((1 + L/W^2)q_1 - q_2/W^2) + (q_2 - Lq_1)\bar{\rho}\rho + q_3\bar{\rho} + q_4\rho \right),$$

which means that the Grassmann part of the operator acts on a vector $q = (q_1, q_2, q_3, q_4) \in \mathbb{C}^4$ as 4×4 matrix $W^2 \mathcal{Q}$ with a block matrix

$$\mathcal{Q} = \begin{pmatrix} \check{\mathcal{Q}} & 0 \\ 0 & I \end{pmatrix},$$

where the 2×2 matrix $\check{\mathcal{Q}}$ has the form

$$\check{\mathcal{Q}} = \begin{pmatrix} 1 + L/W^2 & -1/W^2 \\ -L & 1 \end{pmatrix}$$

with the function L of (2.10).

Note that the usual \mathbb{C}^4 inner product of vectors $q = \{q_i\}_{i=1}^4$ and $p = \{p_i\}_{i=1}^4$ does not coincide with the product of two corresponding Grassmann polynomials $q_1 + q_2 \bar{\rho} + q_3 \bar{\rho} + q_4 \rho$, $p_1 + p_2 \bar{\rho} + p_3 \bar{\rho} + p_4 \rho$. For instance, the product of $q_1 + q_2 \bar{\rho}$, $p_1 + p_2 \bar{\rho}$ is obtained by the usual inner product of $(q_1, q_2, 0, 0)$ and $(p_2, p_1, 0, 0)$ (not $(p_1, p_2, 0, 0)$).

Introduce compact integral operators A and A_1 in $L_2[\mathbb{R}]$ with the kernels

$$A(a_1, a_2) = \mathcal{F}_0(a_1)B(a_1, a_2)\mathcal{F}_0(a_2), \quad \mathcal{F}_0(a) = e^{-f_a(a)/2}; \quad (3.1)$$

$$A_1(b_1, b_2) = \mathcal{F}_1(b_1)B(b_1, b_2)\mathcal{F}_1(b_2), \quad \mathcal{F}_1(b) = e^{-f_b(b)/2}; \quad (3.2)$$

$$B(a_1, a_2) = (2\pi)^{-1/2} W e^{-W^2(a_1 - a_2)^2/2},$$

where f_a and f_b are defined in (2.9). Denote also

$$K = A \otimes A_1, \quad \check{K} = \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}, \quad \check{K}_1 = \begin{pmatrix} \check{K} & 0 \\ 0 & \check{K} \end{pmatrix}. \quad (3.3)$$

Let \mathcal{B} be the operator of multiplication by

$$d(b) = b + i\sqrt{4 - E^2}/2.$$

Set

$$\check{\mathcal{B}} = \begin{pmatrix} \mathcal{B} & 0 \\ 0 & \mathcal{B} \end{pmatrix}, \quad \check{\mathcal{B}}_1 = \begin{pmatrix} \check{\mathcal{B}} & 0 \\ 0 & \check{\mathcal{B}} \end{pmatrix}, \quad (3.4)$$

$$e'_1 = (1, 0, 0, 0)^t, \quad e'(L) = (-\bar{L}, 1, 0, 0)^t,$$

$$e_1 = (1, 0)^t, \quad e_2 = (0, 1)^t, \quad e(L) = (-\bar{L}, 1)^t, \quad e_L = (1, -L/W)^t.$$

With these notations we can rewrite (2.11) as

$$\begin{aligned} \bar{g}_n(E) &= \frac{1}{n} \sum_{j=0}^{n-1} \left(((\check{K}_1 \mathcal{Q})^j \check{\mathcal{B}}_1 (\check{K}_1 \mathcal{Q})^{n-1-j} e'_1, e'(L))_4 \mathcal{F}, \bar{\mathcal{F}} \right) \\ &= \frac{1}{n} \sum_{j=0}^{n-1} \left(((\check{K} \check{\mathcal{Q}})^j \check{\mathcal{B}} (\check{K} \check{\mathcal{Q}})^{n-1-j} e_1, e(L))_2 \mathcal{F}, \bar{\mathcal{F}} \right) \end{aligned} \quad (3.5)$$

where $(\cdot, \cdot)_4$ and $(\cdot, \cdot)_2$ mean inner products in \mathbb{C}^4 and \mathbb{C}^2 respectively,

$$\mathcal{F}(a, b) = \mathcal{F}_0(a)\mathcal{F}_1(b),$$

and we have used the block-diagonal structure of \check{K}_1 , \check{B}_1 , and \mathcal{Q} .

To study the r.h.s. of (3.5), let us rewrite it in a more convenient form. Introduce the matrices

$$T = \begin{pmatrix} 0 & W^{-1/2} \\ W^{1/2} & 0 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & -L/W \\ -1/W & 1 + L/W^2 \end{pmatrix}. \quad (3.6)$$

Note that for $|E| > \varepsilon_*$ with any fixed $\varepsilon_* > 0$

$$|L(a_j, b_j)| < L_0, \quad a_j \in \mathbb{R}, \quad b_j \in \mathbb{R}, \quad (3.7)$$

for some constant L_0 depending on E , and hence

$$\|S\| \leq 1 + C_0/W, \quad (3.8)$$

where one can take $C_0 = 2 + L_0$.

Remark 3.1 *In order to have the bound (3.7) valid for any E , in the case of $|E| \leq \varepsilon_*$ with sufficiently small $\varepsilon_* > 0$ we deform the contour of integration with respect to a in some small neighbourhood U_ε of $a = 0$ in such a way that guarantees the conditions*

$$\sup_{a \in U_\varepsilon} \{|\cos^{-1/2} 2\phi(a)|\} \sup_{a \in U_\varepsilon} \{e^{-\Re f_a(a)/2}\} \leq 1 - \delta, \quad (3.9)$$

where $\phi(a)$ is an angle between the contour and the real line at the point a and $\delta > 0$ is some fixed number which can be chosen as small as we want (see Section 4). The above condition will be important below (see the proof of (5.15)).

It is easy to check that

$$\check{Q} = TST^{-1}.$$

Hence, since

$$T^{-1}\check{K}T = \check{K}, \quad T^{-1}\check{B}T = \check{B},$$

we have

$$\begin{aligned} \left(((\check{K}\check{Q})^j \check{B}(\check{K}\check{Q})^{n-1-j} e_1, e(L))_2 \mathcal{F}, \bar{\mathcal{F}} \right) &= \left(((\check{K}S)^j \check{B}(\check{K}S)^{n-1-j} T^{-1} e_1, T^* e(L))_2 \mathcal{F}, \bar{\mathcal{F}} \right) \\ &= W \left(((\check{K}S)^j \check{B}(\check{K}S)^{n-1-j} e_2, e_L)_2 \mathcal{F}, \bar{\mathcal{F}} \right) \end{aligned}$$

since

$$T^{-1}e_1 = W^{1/2}(0, 1)^t = W^{1/2}e_2, \quad T^*e(L) = W^{1/2}(1, -L/W)^t = W^{1/2}e_L$$

Thus, denoting

$$\mathcal{K} = \check{K}S \quad (3.10)$$

we get finally from (3.5)

$$\bar{g}_n(E) = \frac{W}{n} \sum_{j=0}^{n-1} \left((\mathcal{K}^j \check{B} \mathcal{K}^{n-1-j} e_2, e_L)_2 \mathcal{F}, \bar{\mathcal{F}} \right), \quad (3.11)$$

In what follows it will be important for us that the same argument applied to (2.12) implies

$$1 = W \left((\mathcal{K}^{n-1} e_2, e_L)_2 \mathcal{F}, \bar{\mathcal{F}} \right). \quad (3.12)$$

4 Analysis of \mathcal{K}

In this section we apply the method developed in [17], based on the proposition, which is a standard linear algebra tool

Proposition 4.1 *Given a compact operator \mathcal{K} , assume that there is an orthonormal basis $\{\Psi_l\}_{l \geq 0}$ such that the resolvent*

$$\widehat{\mathcal{G}}_{jk}(z) = (\widehat{\mathcal{K}} - z)_{jk}^{-1}, \quad \widehat{\mathcal{K}} = \{\mathcal{K}_{jk}\}_{j,k=1}^{\infty}$$

is uniformly bounded in $z \in \Omega \subset \mathbb{C}$, where Ω is some domain. Then

(i) the eigenvalues of \mathcal{K} in Ω coincide with zeros of the function

$$F(z) := \mathcal{K}_{00} - z - (\widehat{\mathcal{G}}(z)\kappa, \kappa^*), \\ \kappa = (\mathcal{K}_{10}, \mathcal{K}_{20}, \dots), \quad \kappa^* = (\mathcal{K}_{10}^*, \mathcal{K}_{20}^*, \dots);$$

(ii) if $F(z)$ has a unique root z_ in Ω (and $F'(z_*) \neq 0$), then the resolvent $\mathcal{G}(z) = (\mathcal{K} - z)^{-1}$ can be represented in the form*

$$\mathcal{G}_{ij}(z) = \frac{\eta_i \bar{\eta}_j^*}{F'(z_*)(z_* - z)} + R_{ij}(z), \quad \eta_i = \eta_i(z_*), \quad \eta_j^* = \eta_j^*(z_*), \quad (4.1)$$

$$\eta_i(z) = \delta_{0i} - (1 - \delta_{0i})(\widehat{\mathcal{G}}(z)\kappa)_i, \quad \eta_j^*(z) = \delta_{0j} - (1 - \delta_{0j})(\widehat{\mathcal{G}}^*(z)\kappa^*)_j, \quad (4.2)$$

where $R_{ij}(z)$ is an analytic in Ω matrix-function whose norm satisfies the bound

$$\|R\| \leq \sup_{z \in \Omega} \|\widehat{\mathcal{G}}\| + \sup_{z \in \Omega} \left| \frac{d}{dz} \frac{z - z_*}{F(z)} \right| (1 + \|G\| \cdot \|\kappa\|)(1 + \|G\| \cdot \|\kappa^*\|) \\ + \sup_{z \in \Omega} \left| \frac{z - z_*}{F(z)} \right| (1 + \|\widehat{\mathcal{G}}\|)(1 + \|G\| \cdot \|\kappa\|)(1 + \|G\| \cdot \|\kappa^*\|). \quad (4.3)$$

Proof. The assertion (i) follows from the standard Schur inversion formula valid for any $z : F(z) \neq 0$:

$$\mathcal{G}_{ij}(z) = \widehat{\mathcal{G}}_{ij}(z) + \frac{\eta_i(z)\eta_j^*(z)}{F(z)}, \quad (4.4)$$

where we set $\widehat{\mathcal{G}}_{ij}(z) = 0$ if $i = 0$ or $j = 0$.

To prove the assertion (ii), we write

$$R_{ij}(z) = \widehat{\mathcal{G}}_{ij}(z) + \frac{\eta_i(z)\eta_j^*(z)}{F(z)} - \frac{\eta_i(z_*)\eta_j^*(z_*)}{F'(z_*)(z - z_*)},$$

and use

$$\left| \frac{\eta_i(z)\eta_j^*(z)(z - z_*)}{F(z)} - \frac{\eta_i(z_*)\eta_j^*(z_*)}{F'(z_*)} \right| \leq |z - z_*| \cdot \sup_{z \in \Omega} \left| \frac{d}{dz} \left(\frac{\eta_i(z)\eta_j^*(z)(z - z_*)}{F(z)} \right) \right|$$

and

$$\widehat{\mathcal{G}}'(z) = \widehat{\mathcal{G}}^2(z)$$

to obtain the bound (4.3). \square

Analysis of spectral properties of \mathcal{K} is based on the analysis of K of (3.3). Recall the definitions (3.1), (3.2) and choose W, n -independent $\delta > 0$, which is small enough to provide that the domain $\omega_\delta = \{x \in \mathbb{R} : |\mathcal{F}_0(x)| > 1 - \delta\}$ contains two non intersecting sub domains $\omega_\delta^+, \omega_\delta^-$, such that each of $\omega_\delta^+, \omega_\delta^-$ contains one of the points $x = a_+$ and $x = a_-$ of maximum $\mathcal{F}_0(x)$ (easier speaking, $\omega_\delta^+, \omega_\delta^-$ are two non-intersecting neighbourhood of points a_+ and a_- (see Lemma 2.1)). Set also $\omega_{1,\delta} = \{x \in \mathbb{R} : |\mathcal{F}_1(x)| > 1 - \delta\}$.

To choose a convenient basis in $L_2[\mathbb{R}]$, take $c_* : \Re c_* > 0$, set

$$\alpha = \sqrt{\frac{c_*}{2}} \left(1 + \frac{c_*}{2W^2}\right)^{1/2} =: \alpha_1 + i\alpha_2,$$

and consider the system of the functions

$$\begin{aligned} \psi_0(x) &= e^{-\alpha W x^2} \sqrt[4]{\alpha W / \pi}; \\ \psi_k(x) &= h_k^{-1/2} e^{-\alpha W x^2} e^{2\alpha_1 W x^2} \left(\frac{d}{dx}\right)^k e^{-2\alpha_1 W x^2} = e^{-\alpha W x^2} p_k(x); \\ h_k &= k! (4\alpha_1 W)^{k-1/2} \sqrt{2\pi}, \quad k = 1, 2, \dots \end{aligned} \tag{4.5}$$

It is easy to see that p_k is the k th polynomials, orthogonal with the weight $e^{-2\alpha_1 W x^2}$ (it is the k th Hermite polynomial of $x\sqrt{2\alpha_1 W}$ with a proper normalization).

Now let $\{\psi_k\}_{k=0}^\infty$ be (4.5) with $c_* = c_+$ of (2.15). Consider the set $\{\psi_{k,\delta}^+\}$ obtained by the Gramm-Schmidt orthonormalization procedure of

$$\psi_k^+(x) = \psi_k(x - a_+)$$

on ω_δ^+ . Since $\psi_{k,\delta}^+(x) = O(e^{-cW})$ for $x \notin \omega_\delta^+$, one can obtain easily

$$\psi_{k,\delta}^+(x) = \psi_k^+(x) + O(e^{-cW}), \quad k \ll W.$$

By the same way we construct $\{\psi_k'(x)\}_{k=0}^\infty$ and $\{\psi_{k,\delta}'(x)\}_{k=0}^\infty$ on $\omega_{1,\delta}$ (with b_s instead of a_+), and $\{\psi_k^-(x)\}_{k=0}^\infty$ and $\{\psi_{k,\delta}^-(x)\}_{k=0}^\infty$ on ω_δ^- (with $c_* = c_-$ and a_- instead of a_+). Everywhere below we take

$$m = [\log^2 W]$$

and consider two vector systems

$$\begin{aligned} \{\Psi_{\bar{k}}^+(a, b)\}_{|k| \leq m} &= \{\psi_{k_1,\delta}^+(a) \psi_{k_2,\delta}'(b)\}_{|k| \leq m}, \\ \{\Psi_{\bar{k}}^-(a, b)\}_{|k| \leq m} &= \{\psi_{k_1,\delta}^-(a) \psi_{k_2,\delta}'(b)\}_{|k| \leq m}. \end{aligned} \tag{4.6}$$

Denote P^+ and P^- the projections on the subspaces spanned on the systems $\{\Psi_{\bar{k},\delta}^+\}_{|k| \leq m}$ and $\{\Psi_{\bar{k},\delta}^-\}_{|k| \leq m}$ respectively. Evidently these projection operators are orthogonal to each other. Set

$$P = P^+ + P^-, \quad \mathcal{L}_1 = P\mathcal{H}, \quad \mathcal{L}_2 = (1 - P)\mathcal{H}, \quad \mathcal{H} = \mathcal{L}_1 \oplus \mathcal{L}_2, \tag{4.7}$$

where $\mathcal{H} = L_2(\mathbb{R}^2)$.

Note also that for any u supported in some domain Ω and any $C > 0$

$$(Ku)(a, b) = O(e^{-cW^2}) \text{ for } (a, b) : \text{dist}\{(a, b), \Omega\} \geq C > 0. \tag{4.8}$$

Now consider the operator K as a block operator with respect to the decomposition (4.7). It has the form

$$\begin{aligned} K^{(11)} &= K^+ \oplus K^- + O(e^{-cW^2}), \\ K^+ &:= P^+ K P^+, \quad K^- = P^- K P^-, \\ K^{(12)} &= P^+ K (I^+ - P^+) \oplus P^- K (I^- - P^-) + O(e^{-cW}), \\ K^{(21)} &= (I^+ - P^+) K P^+ \oplus (I^- - P^-) K P^- + O(e^{-cW}), \end{aligned} \quad (4.9)$$

where I^+ and I^- are the operator of the multiplication by $1_{\omega_\delta^+} 1_{\omega_{1,\delta}}$ and $1_{\omega_\delta^-} 1_{\omega_{1,\delta}}$ respectively. Indeed, (4.8) and

$$(A\psi_k^+)(x) = O(e^{-cW}) \text{ for } |x - a_+| \geq C > 0,$$

yield $P^+ K P^- f = O(e^{-cW^2})$, $P_- K (I^+ - P^+) f = O(e^{-cW})$, etc.

Let \hat{K} be K without the line and the column, corresponding to Ψ_0^+ , and \hat{K}^+ , is defined similarly. Denote also

$$\kappa^+ = K \Psi_0^+ - (K \Psi_0^+, \Psi_0^+) \Psi_0^+, \quad \kappa_*^+ = K^* \Psi_0^+ - (K \Psi_0^+, \Psi_0^+) \Psi_0^+,$$

and set

$$\alpha_+ = \sqrt{\frac{c_+}{2}} \left(1 + \frac{c_+}{2W^2}\right)^{1/2} =: \alpha_1 + i\alpha_2. \quad (4.10)$$

$$\lambda_{0,+} = \left(1 + \frac{2\alpha_+}{W} + \frac{c_+}{W^2}\right)^{-1/2}. \quad (4.11)$$

Theorem 4.1 *Given an operator K of the form (3.3), we have*

$$\left| |\lambda_0(K)| - |\lambda_{0,+}|^2 \right| \leq CW^{-3/2}.$$

Moreover, for any z satisfying conditions

$$1 - \frac{5\alpha_1}{2W} < |z| \leq 1 + \frac{C_0 + 5\alpha_1/2}{W}, \quad |z - |\lambda_{0,+}|^2| \geq c/W. \quad (4.12)$$

with C_0 of (3.8) we have

$$\|(\hat{K}^{(11)} - z)^{-1}\| \leq CW, \quad (4.13)$$

$$\|\hat{K}^{(12)}\| \leq Cm/W, \quad \|\hat{K}^{(21)}\| \leq Cm^{3/2}/W^{3/2}, \quad (4.14)$$

$$\|K^{(22)}\| \leq 1 - Cm^{1/3}/W. \quad (4.15)$$

In addition,

$$\|\kappa^+\| \leq C/W^{3/2}, \quad \|\kappa_*^+\| \leq C/W, \quad (4.16)$$

$$(K \Psi_0^+, \Psi_0^+) = \lambda_{0,+}^2 + O(W^{-3/2}), \quad (4.17)$$

and there is $0 < q < 1$ such that for all $|k|, |k'| \leq m$

$$\begin{aligned} (P^+ \hat{G} P^+)_{\bar{k}, \bar{k}'} &= W G_{\bar{k}, \bar{k}'}^{(ev)} + W^{1/2} G_{\bar{k}, \bar{k}'}^{(r)}, \\ |G_{\bar{k}, \bar{k}'}^{(ev)}| + |G_{\bar{k}, \bar{k}'}^{(r)}| &\leq Cq^{|\bar{k} - \bar{k}'|/2}, \quad G_{\bar{k}, \bar{k}'}^{(ev)} = 0, \text{ if } \bar{k} - \bar{k}' \notin 2\mathbb{Z}^2. \end{aligned} \quad (4.18)$$

the vectors η and η^* defined as in (4.1) satisfy the conditions

$$\begin{aligned}\eta &= \Psi_0^+ + W^{-1/2} \tilde{\eta}, \quad |\tilde{\eta}_{\bar{k}}| \leq Cq^{|\bar{k}|/2}, \\ \eta^* &= \eta^{*(ev)} + W^{-1/2} \tilde{\eta}^*, \quad |\eta_{\bar{k}}^{*(ev)}| + |\tilde{\eta}_{\bar{k}}^*| \leq Cq^{|\bar{k}|/2}, \quad \eta_{\bar{k}}^{*(ev)} = 0, \text{ if } \bar{k} \notin 2\mathbb{Z}^2.\end{aligned}\tag{4.19}$$

Defer the proof of Theorem 4.1 to the next section and continue the analysis of \mathcal{K} . Write \mathcal{K} as

$$\mathcal{K} = \begin{pmatrix} K^{(11)}S & K^{(12)}S \\ K^{(21)}S & K^{(22)}S \end{pmatrix},\tag{4.20}$$

where

$$K^{(\alpha\alpha')}S = \begin{pmatrix} K^{(\alpha\alpha')}S_{11} & K^{(\alpha\alpha')}S_{12} \\ K^{(\alpha\alpha')}S_{21} & K^{(\alpha\alpha')}S_{22} \end{pmatrix}.$$

Since all vectors in $\{\Psi_{\bar{k}}^+\}_{|k| \leq m}$ and $\{\Psi_{\bar{k}}^-\}_{|k| \leq m}$ possess the property

$$\begin{aligned}|\Psi_{\bar{k}}^+(a, b)| &\leq e^{-c \log^2 W}, \quad \text{if } |a - a_+| + |b - b_s| \geq CW^{-1/2} \log W, \\ |\Psi_{\bar{k}}^-(a, b)| &\leq e^{-c \log^2 W}, \quad \text{if } |a - a_-| + |b - b_s| \geq CW^{-1/2} \log W\end{aligned}$$

for sufficiently big $C > 0$, we have

$$|S - S^+| \Psi_{\bar{k}}^+(a, b) = O(W^{-3/2} \log W), \quad |S - S^-| \Psi_{\bar{k}}^-(a, b) = O(W^{-3/2} \log W),$$

where S^+ and S^- have the form (3.6) with L replaced by L^+ and L^- of (2.16) respectively. Hence

$$K^{(11)}S = \begin{pmatrix} K^+S^+ + O(W^{-3/2} \log W) & O(e^{-cW^2}) \\ O(e^{-cW^2}) & K^-S^- + O(W^{-3/2} \log W) \end{pmatrix}.$$

It is useful to rewrite \mathcal{K} in a more convenient form. Write

$$S^+ = V\Lambda V^{-1}, \quad \Lambda = \begin{pmatrix} \lambda_1^+ & 0 \\ 0 & \lambda_2^+ \end{pmatrix}, \quad V = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}$$

where λ_1^+ and λ_2^+ are eigenvalues of S^+ :

$$\lambda_1^+ = 1 + \frac{L^+}{2W^2} + \sqrt{\frac{L^+}{W^2} + \frac{(L^+)^2}{4W^4}}, \quad \lambda_2^+ = 1/\lambda_1^+,\tag{4.21}$$

and V is a 2×2 matrix diagonalizing S^+ . It is easy to check that the eigenvectors of S^+ have the form $(v_{11}, v_{21}) = (-\sqrt{L^+}, 1) + O(W^{-1})$, $(v_{12}, v_{22}) = (\sqrt{L^+}, 1) + O(W^{-1})$, hence

$$\|V\| \leq C, \quad \|V^{-1}\| \leq C.\tag{4.22}$$

Introduce the matrix

$$\check{V} = \begin{pmatrix} V & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix},$$

where the first block corresponds to K^+S^+ , the second one to K^-S^- , and the third one to $K^{(22)}S$. Now set

$$\mathcal{K}_V = \check{V}^{-1} \mathcal{K} \check{V} = \begin{pmatrix} K^+ \Lambda + O\left(\frac{\log W}{W^{3/2}}\right) & O(e^{-cW^2}) & K^{+(12)} V^{-1} S \\ O(e^{-cW^2}) & K^- S^- + O\left(\frac{\log W}{W^{3/2}}\right) & K^{-(12)} S \\ K^{+(21)} S V & K^{-(21)} S & K^{(22)} S \end{pmatrix}. \quad (4.23)$$

Note that

$$K^- S = K^- S^- + O(W^{-3/2} \log W) = \begin{pmatrix} K^- & 0 \\ -K^-/W & K^- \end{pmatrix} + O(W^{-3/2} \log W). \quad (4.24)$$

Then (3.11) and (3.12) can be written as

$$\bar{g}_n(E) = \frac{W}{n} \sum_{j=0}^{n-1} \left((\mathcal{K}_V^j \check{\mathcal{B}} \mathcal{K}_V^{n-1-j} e_V, e_V^*)_2 \mathcal{F}, \bar{\mathcal{F}} \right), \quad e_V = V^{-1} e_2, \quad e_V^* = V^* e_L, \quad (4.25)$$

$$1 = W \left((\mathcal{K}_V^{n-1} e_V, e_V^*)_2 \mathcal{F}, \bar{\mathcal{F}} \right). \quad (4.26)$$

Now we can formulate the main result of the section

Theorem 4.2 *Given an operator \mathcal{K}_V defined in (4.23) we have*

$$\lambda_0(\mathcal{K}_V) = 1, \quad |\lambda_1(\mathcal{K}_V)| \leq 1 - c/W. \quad (4.27)$$

Moreover, the resolvent $\mathcal{G}(z) = (\mathcal{K}_V - z)^{-1}$ can be written as

$$\mathcal{G}(z) = \frac{P_\eta}{1-z} + R(z), \quad \|R\| \leq CW, \quad \text{if } z \in \Omega_0 = \{|z| \geq 1 - c/2W\}, \quad (4.28)$$

where P_η is a rank one operator of the form

$$P_\eta = \eta_V \otimes \eta_V^*, \quad \eta_V = \eta \otimes e_1 + O(W^{-1}), \quad \eta_V^* = \eta^* \otimes e_1 + O(W^{-1}), \quad (4.29)$$

with η, η^* of (4.19).

Proof. Prove first that

$$|\lambda_0(\mathcal{K}_V) - 1| \leq C \log^2 W / W^{3/2}. \quad (4.30)$$

In order to apply Proposition 4.1, we want to prove first that for $\hat{\mathcal{G}}(z) = (\hat{\mathcal{K}}_V - z)^{-1}$

$$\|\hat{\mathcal{G}}(z)\| \leq CmW, \quad \text{if } z \in \Omega = \{1 - \alpha_1/3W \leq |z| \leq 1 + C_0/W\} \quad (4.31)$$

with α_1, C_0 of (4.12).

According to (4.23) and the formula for the inverse of the block matrix, it is easy to see that to prove (4.31) it suffices to check

$$\begin{aligned} \|(\hat{K}^+ \lambda_1^+ - z)^{-1}\| &\leq CW, \quad \|(K^+ \lambda_2^+ - z)^{-1}\| \leq CW, \quad \|(K^- S^- - z)^{-1}\| \leq CW, \\ \|K^{+(12)} V^{-1} S\| &\leq Cm/W, \quad \|K^{-(12)} S\| \leq Cm/W, \\ \|K^{+(21)} S V\| &\leq C(m/W)^{3/2}, \quad \|K^{-(21)} S\| \leq C(m/W)^{3/2}, \\ \|(K^{(22)} S - z)^{-1}\| &\leq CW/m^{1/3} \end{aligned} \quad (4.32)$$

for $z \in \Omega$. Rewrite

$$(\hat{K}^+ \lambda_1^+ - z)^{-1} = (\lambda_1^+)^{-1} (\hat{K}^+ - z/\lambda_1^+)^{-1}, \quad (\hat{K}^+ \lambda_2^+ - z)^{-1} = \lambda_1^+ (\hat{K}^+ - z/\lambda_2^+)^{-1}.$$

Note that according to (2.16) and (4.21)

$$\lambda_1^+ = \lambda_{0,+}^{-2}.$$

Now using (4.11) it is easy to check that if $z \in \Omega$ of (4.31), then both z/λ_1^+ and z/λ_2^+ satisfy (4.12), and therefore we get the first line of (4.32). The second and the third lines follow from (4.13) – (4.14), (4.24) and (3.8), (4.22). Moreover, (3.8) and the representation (4.20) combined with (4.14) – (4.15) yield

$$\|K^{(22)}S\| \leq 1 - \frac{Cm^{1/3}}{W}.$$

Hence

$$\|(K^{(22)}S - z)^{-1}\| \leq \frac{1}{|z|} \sum_{s=0}^{\infty} \frac{\|K^{(22)}S\|^s}{|z|^s} \leq C_1 W/m^{1/3}, \quad (4.33)$$

which finishes the proof of (4.32), thus (4.31).

Now we can apply Proposition 4.1 to \mathcal{K}_V and consider

$$\begin{aligned} F(z) &= \lambda_1^+ K_{00}^+ - z - (\hat{\mathcal{G}}(z) \kappa_V, \kappa_V^*) + O\left(\frac{\log W}{W^{3/2}}\right) \\ &= \lambda_{0,+}^2 \lambda_1^+ - z - (\hat{\mathcal{G}}(z) \kappa_V, \kappa_V^*) + O\left(\frac{\log W}{W^{3/2}}\right), \end{aligned}$$

where κ_V and κ_V^* are the column and the line of \mathcal{K}_V that correspond to the vector $\Psi_{\bar{0}} \otimes e_1$. To get the second equality here we used (4.17). By (4.16) and (4.22)

$$\|\kappa_V^*\| \leq CW^{-1}, \quad \|\kappa_V\| \leq CW^{-3/2}. \quad (4.34)$$

Define also (recall $\lambda_{0,+}^2 \lambda_1^+ = 1$)

$$F_0(z) = \lambda_{0,+}^2 \lambda_1^+ - z = 1 - z, \quad \sigma = \{z : |z - 1| \leq C_* W^{-3/2} \log^2 W\}$$

with sufficiently big C_* . Then (4.34) and (4.31) yield

$$|(\hat{\mathcal{G}}(z) \kappa_V, \kappa_V^*)| \leq Cm/W^{3/2} \Rightarrow |F_0(z)| > |F(z) - F_0(z)|, \quad z \in \partial\sigma,$$

and hence the Rouché theorem implies that $F(z)$, $F_0(z)$ have the same number of zeros (one) in σ , which gives (4.30). Taking $z \in \sigma(z_0) = \{z \in \mathbb{C} : |z - z_0| \leq C_* \log^2 W/W^{3/2}\}$ for any $|z_0 - \lambda_0(\mathcal{K}_V)| > 2C_*/W$ satisfying (4.28), by the same argument one can obtain that $F(z)$ has the same number of zeros as $F_0(z)$ in $\sigma(z_0)$ (i.e. zero, since $\lambda_{0,+}^2 \lambda_2^+$, $|\lambda_{0,+}|^2$ do not satisfy (4.28)), which implies the second bound of (4.27) with $c = \alpha_1/3$. The representation (4.28) follows from (4.1)-(4.3) if we take into account (4.19), (4.31), (4.34) and the fact that

$$\kappa_V = \lambda_1^+ \kappa \otimes e_1 + O(W^{-3/2}).$$

Let us prove now the first relation in (4.27). The Cauchy formula for the resolvent yields

$$\mathcal{K}_V^{n-1} = \frac{1}{2\pi i} \oint_L z^{n-1} \mathcal{G}(z) dz$$

for any closed contour L which contains all eigenvalues of \mathcal{K} . Let us choose L as a union of two circles:

$$L_1 = \{z : |z| = 1 - c/2W\}, \quad L_0 = \{z : |z - 1| \leq c/3W\}$$

with some sufficiently small but n, W -independent c . Hence we get

$$\begin{aligned} W \left((\mathcal{K}_V^{n-1} e_V, e_V^*)_2 \mathcal{F}, \bar{\mathcal{F}} \right) &= \frac{W}{2\pi i} \left(\oint_{L_0} + \oint_{L_1} \right) z^{n-1} \left((\mathcal{G}(z) e_V, e_V^*)_2 \mathcal{F}, \bar{\mathcal{F}} \right) \\ &= \frac{W}{2\pi i} \oint_{L_0} z^{n-1} dz \left((\mathcal{G}(z) e_V, e_V^*)_2 \mathcal{F}, \bar{\mathcal{F}} \right) + O(W^2 e^{-cn/W}) =: I_1^0 + O(mW^2 e^{-cn/W}), \end{aligned}$$

where the bound for the remainder follows from (4.22) and the bound on the norm of the resolvent on the contour L_1 obtained from (4.28):

$$\|\mathcal{G}(z)\| \leq CmW, \quad z \in L. \quad (4.35)$$

But using the representation (4.28) – (4.29), definition of e_V, e_V^* in (4.25), and the Cauchy theorem we have

$$I_1^0 = W \lambda_0^{n-1}(\mathcal{K}_V)(\tilde{\eta}_V, \bar{\mathcal{F}})(\tilde{\eta}_V^*, \bar{\mathcal{F}}),$$

where

$$\tilde{\eta}_V = (V^{-1})_{12} \eta \otimes e_1 + O(W^{-1}), \quad \tilde{\eta}_V^* = (V^* e_L)_1 \eta^* \otimes e_1 + O(W^{-1}) \quad (4.36)$$

with η, η^* of (4.19). Hence, on the basis of (4.26) we conclude that

$$1 = W \lambda_0^{n-1}(\mathcal{K}_V)(\tilde{\eta}_V, \bar{\mathcal{F}})(\tilde{\eta}_V^*, \bar{\mathcal{F}}) + O(e^{-cn/2W}).$$

Using that $\lambda_0(\mathcal{K}_V)$ and $(\tilde{\eta}_V, \bar{\mathcal{F}})(\tilde{\eta}_V^*, \bar{\mathcal{F}})$ do not depend on n , and n in the above formula can be taken arbitrary large, we conclude that

$$\lambda_0(\mathcal{K}_V) = 1, \quad W(\tilde{\eta}_V, \bar{\mathcal{F}})(\tilde{\eta}_V^*, \bar{\mathcal{F}}) = 1, \quad (4.37)$$

which gives the first equality in (4.27). \square

Proof of Theorem 1.1. Set

$$\mathcal{B}^\circ = \mathcal{B} - (E/2 + i\sqrt{4 - E^2}/2)I \quad (4.38)$$

Evidently

$$\sum_{j=0}^{n-1} \mathcal{K}_V^j \mathcal{B}^\circ \mathcal{K}_V^{n-1-j} = \frac{1}{(2\pi i)^2} \oint_L dz_1 \oint_{L'} dz_2 \frac{z_1^n - z_2^n}{z_1 - z_2} \mathcal{G}(z_1) \mathcal{B}^\circ \mathcal{G}(z_2),$$

where the contour L was chosen above and the contour $L' = L'_0 \cup L'_1$ is chosen similarly, but on the distance d/W from L with some sufficiently small fixed d . Then by (3.11) and the above formula we obtain

$$\bar{g}_n(E) = n^{-1}(I_1 + I_2 + I_3 + I_4),$$

where I_1 corresponds to the integral over $z_1 \in L_0, z_2 \in L'_0$, I_2 corresponds to the integral over $z_1 \in L_0, z_2 \in L'_1$, I_3 corresponds to the integral over $z_1 \in L_1, z_2 \in L'_0$, and I_4 corresponds to the integral over $z_1 \in L_1, z_2 \in L'_1$. The bound for the resolvent (4.35), and the estimates

$$\begin{aligned} \|[\mathcal{B}^\circ, \mathcal{G}(z_2)]\| &= \|\mathcal{G}(z_2)[\mathcal{B}^\circ, \mathcal{K}_V]\mathcal{G}(z_2)\| \leq Cm^2W \\ \Rightarrow \|\mathcal{B}^\circ \mathcal{G}(z_2) \mathcal{F}\| &\leq \|\mathcal{G}(z_2) \mathcal{B}^\circ \mathcal{F}\| + \|[\mathcal{B}^\circ, \mathcal{G}(z_2)] \mathcal{F}\| \\ &\leq CmW \|\mathcal{B}^\circ \mathcal{F}\| + Cm^2W \|\mathcal{F}\| \leq Cm^2W \end{aligned}$$

yield for some absolute $p > 0$

$$|I_4| = \left| \frac{W}{(2\pi i)^2} \oint_{L_1} dz_1 \oint_{L'_1} dz_2 \frac{z_1^n - z_2^n}{z_1 - z_2} \left((\mathcal{G}(z_1) \mathcal{B}^\circ \mathcal{G}(z_2) e_V, e_V^*)_2 \mathcal{F}, \bar{\mathcal{F}} \right) \right| \leq Cm^p W^4 e^{-nc/W}.$$

Let us prove the bound for I_2 . We write first

$$\begin{aligned} I_2 &= \frac{W}{(2\pi i)^2} \oint_{L_0} dz_1 \oint_{L'_1} dz_2 \frac{z_1^n}{z_1 - z_2} \left((\mathcal{G}(z_1) \mathcal{B}^\circ \mathcal{G}(z_2) e_V, e_V^*)_2 \mathcal{F}, \bar{\mathcal{F}} \right) + O(m^p W^4 e^{-nc/W}) \\ &= \frac{W}{2\pi i} \oint_{L'_1} \frac{dz_2}{1 - z_2} \left((P_\eta \mathcal{B}^\circ \mathcal{G}(z_2) e_V, e_V^*)_2 \mathcal{F}, \bar{\mathcal{F}} \right) + O(m^p W^4 e^{-nc/W}) \\ &= \frac{W}{2\pi i} \lim_{R \rightarrow \infty} \left(\oint_{|z|=R} - \oint_{L'_0} \right) \frac{dz_2}{1 - z_2} \left((P_\eta \mathcal{B}^\circ \mathcal{G}(z_2) e_V, e_V^*)_2 \mathcal{F}, \bar{\mathcal{F}} \right) + O(m^p W^4 e^{-nc/W}) \\ &= -\frac{W}{2\pi i} \oint_{L'_0} \frac{dz_2}{1 - z_2} \left((P_\eta \mathcal{B}^\circ \mathcal{G}(z_2) e_V, e_V^*)_2 \mathcal{F}, \bar{\mathcal{F}} \right) + O(m^p W^4 e^{-nc/W}) \\ &= I'_2 + O(m^p W^4 e^{-nc/W}). \end{aligned}$$

To estimate I'_2 , observe that by (4.4)

$$\begin{aligned} \left((P_\eta \mathcal{B}^\circ \mathcal{G}(z) e_V, e_V^*)_2 \mathcal{F}, \bar{\mathcal{F}} \right) &= C_*(V)(\mathcal{F}, \eta)(\mathcal{B}^\circ G(z) \mathcal{F}, \eta^*) \\ &= C_*(V)(\mathcal{F}, \eta) \left(\frac{(\mathcal{F}, \Psi_{\bar{0}}) - (\mathcal{F}, \hat{G}^* \kappa^*)}{F(z)} \left((\Psi_{\bar{0}}, \mathcal{B}^\circ \eta^*) - (\hat{G} \kappa, \mathcal{B}^\circ \eta^*) \right) + (\mathcal{F}, \hat{G}^* \mathcal{B}^\circ \eta^*) \right), \end{aligned}$$

where $C_*(V)$ is some constant depending on the entries of V , $G = (K - z)^{-1}$, $\hat{G} = (\hat{K} - z)^{-1}$, $\kappa, \kappa^*, \eta, \eta^*$ are defined as in Proposition 4.1. We will prove that

$$|(\mathcal{F}, \Psi_{\bar{0}})| \leq CW^{-1/2}, \quad |(\mathcal{F}, \hat{G} \kappa^*)| \leq CW^{-1/2}, \quad |(\mathcal{F}, \eta)| \leq CW^{-1/2}, \quad (4.39)$$

$$|(\Psi_{\bar{0}}, \mathcal{B}^\circ \eta^*)| \leq CW^{-1}, \quad |(\hat{G} \kappa, \mathcal{B}^\circ \eta^*)| \leq CW^{-1}, \quad (4.40)$$

$$|(\mathcal{F}, \hat{G}^* \mathcal{B}^\circ \eta^*)| \leq CW^{-1/2}. \quad (4.41)$$

Since $|z_2 - 1|^{-1} = cW$ on the contour L'_0 and the length of L'_0 is $2\pi(cW)^{-1}$, this inequalities will give us

$$|I'_2| \leq C \Rightarrow I_2 = O(1).$$

The first inequality of (4.39) can be obtained by the direct calculations. The second and third follow from the bounds (4.19) and (4.18)

$$|\eta_{\bar{k}}| \leq Cq^{|\bar{k}|/2}, \quad |(\hat{G} \kappa^*)_{\bar{k}}| \leq Cq^{|\bar{k}|/2}, \quad |\bar{k}| \leq m.$$

Observe that by (4.18) and (4.19)

$$(\mathcal{B}^\circ \eta^*)(a, b) = (b - b_s) \eta^{*(ev)}(a, b) + W^{-1/2} (b - b_s) \tilde{\eta}^*(a, b) + O(W^{-m/4}),$$

where $\eta^{*(ev)}(a, b)$ contains the sum of $\Psi_{(2k_1, 2k_2)}^+(a, b)$, and $\tilde{\eta}^*(a, b)$ contains the sum of $\Psi_{(k_1, k_2)}(a, b)$ (with any \bar{k}) with exponentially decreasing coefficients. Then the structure of \hat{G} (4.18) implies that

$$\begin{aligned} (\hat{G}^* \mathcal{B}^\circ \eta^*)(a, b) &= W(b - b_s) \nu^{(ev)}(a, b) + W^{1/2} (b - b_s) \tilde{\nu}(a, b) + O(W^{-m/4}), \\ |\nu_{\bar{k}}^{(ev)}| + |\tilde{\nu}_{\bar{k}}^*| &\leq Cq^{|\bar{k}|/2}, \quad (|\bar{k}| \leq m) \end{aligned}$$

where $\nu^{(ev)}(a, b)$ still contains only $\Psi_{\bar{k}}$ with $\bar{k} \in 2\mathbb{Z}^2$. It is easy to see that by (4.6) for $|k| \leq m$ we have

$$\begin{aligned} &\int \mathcal{F}(a, b) (b - b_s) \Psi_{(k_1, 2k_2+1)}^+(a, b) da db \\ &= (2\alpha W)^{1/2} \int e^{fa(a)} p_{k_1}((2\alpha W)^{1/2} (a - a_+)) e^{-\alpha W (a - a_+)^2} da \\ &\quad \times \int e^{fb(b)} (b - b_s) p_{2k_2+1}((2\alpha W)^{1/2} (b - b_s)) e^{-\alpha W (b - b_s)^2} db \\ &= (2\alpha W)^{-1} \int p_{k_1}(a) e^{-a^2/2} (1 + O(a^2/W)) da \\ &\quad \times \int b p_{2k_2+1}(b) e^{-b^2/2} (1 + O(b^2/W^2)) db + O(e^{-c \log^2 W}) = O(|k| W^{-1}), \end{aligned}$$

where $\{p_k\}_{k=0}^\infty$ are normalized Hermit polynomials (with a weight e^{-x^2}).

The same argument applied to $\Psi_{(k_1, 2k_2)}^+$ yields

$$\int \mathcal{F}(a, b) (b - b_s) \Psi_{(k_1, 2k_2)}^+(a, b) da db = O(|k| W^{-5/2}),$$

thus we obtain (4.41). Bounds (4.40) can be obtained similarly. The same argument yields also

$$I_3 = O(1).$$

Now using the identity

$$\frac{1}{(2\pi i)^2} \oint_{L_0} \oint_{L'_0} \frac{z_1^n - z_2^n}{z_1 - z_2} \frac{dz_1 dz_2}{(1 - z_1)(1 - z_2)} = n,$$

the representation (4.1), and the Cauchy theorem, we get

$$\begin{aligned} I_1 &= \frac{W}{(2\pi i)^2} \oint_{L_0} dz_1 \oint_{L'_0} dz_2 \frac{z_1^n - z_2^n}{z_1 - z_2} \left((\mathcal{G}(z_1) \mathcal{B}^\circ \mathcal{G}(z_2) e_V, e_V^*)_2 \mathcal{F}, \bar{\mathcal{F}} \right) \\ &= W n (\mathcal{B}^\circ \eta_V, \eta_V^*) (\tilde{\eta}_V, \bar{\mathcal{F}}) (\tilde{\eta}_V^*, \bar{\mathcal{F}}), \end{aligned}$$

where $\tilde{\eta}_V, \tilde{\eta}_V^*$ are defined in (4.36). Thus, according to (4.37), we have

$$\bar{g}_n(E) = (\mathcal{B}^\circ \eta_V, \eta_V^*) + (E/2 + i\sqrt{4 - E^2}/2) + O(n^{-1}).$$

Hence, to finish the proof of the theorem, it suffices to show that

$$|(\mathcal{B}^\circ \eta, \eta^*)| \leq CW^{-1} \Rightarrow |(\mathcal{B}^\circ \eta_V, \eta_V^*)| \leq CW^{-1}. \quad (4.42)$$

But the first line of (4.19), the definition (4.5) of ψ_k , and the definition (4.38) of \mathcal{B}° yield

$$\eta = \Psi_{\bar{0}} + O(W^{-1/2}) \Rightarrow \mathcal{B}^\circ \eta = cW^{-1/2} \Psi_{\bar{0}1} + O(W^{-1}).$$

Combining this with the second line of (4.19) we obtain (4.42). \square

5 Proof of Theorem 4.1

Let us first introduce the “model” operator

$$A_*^{(c_*)}(x, y) = \mathcal{F}_*(x) B(x, y) \mathcal{F}_*(y), \quad \mathcal{F}_*(x) = e^{-c_* x^2/2}, \quad \Re c_* > 0.$$

The main properties of A_* are given by the following lemma, proved in [17] (see Lemma 3.1):

Lemma 5.1 *Given an orthonormal system $\{\psi_k\}_{k \geq 0}$ defined in (4.5), we have*

$$A_*^{(c_*)} \psi_0 = \lambda_0^{(c_*)} \psi_0, \quad \lambda_0^{(c_*)} = \left(1 + \frac{2\alpha}{W} + \frac{c_*}{W^2}\right)^{-1/2}.$$

*The matrix $A_{*jk}^{(c_*)} := (A_*^{(c_*)} \psi_k, \psi_j)$ is upper triangular, $(A_*^{(c_*)})_{jk} = 0$, if j and k have different evenness, and*

$$A_{*kk}^{(c_*)} = (\lambda_0^{(c_*)})^{2k+1}, \quad A_{*k,k+2}^{(c_*)} = -2i\alpha_2 \frac{\sqrt{(k+1)(k+2)}}{W} \left(1 + O\left(\frac{k+1}{W}\right)\right), \quad (5.1)$$

$$|A_{*k,k+2p}^{(c_*)}| \leq \frac{C^p (k+1)^p}{W^p}. \quad (5.2)$$

In addition, if $\{\tilde{\psi}_k\}$ are defined by (4.5) with c_ replaced by some $c_0 > 0$, and \tilde{P}_l is a projection on the space spanned on $\{\tilde{\psi}_r\}_{r=0}^l$, and P_m is a similar projection for $\{\psi_k\}_{k=0}^m$, then*

$$\|\tilde{P}_l(1 - P_m)\| \leq Cl^3/m. \quad (5.3)$$

Recall that $m = \lfloor \log^2 W \rfloor$, and thus $\psi_{k,\delta}^+(y)$ is $O(e^{-c \log^2 W})$ for $|y - a_+| \geq 2CW^{-1/2} \log W$ (for sufficiently big $C > 0$ and $k \leq m$). Therefore, we have

$$A\psi_{k,\delta}^+(x) = O(e^{-c \log^2 W}), \quad |x - a_+| \geq CW^{-1/2} \log W,$$

where A is defined in (3.1). In addition, $A\psi_{k,\delta}^+(x)$ can be written in the form ($k < m$)

$$A\psi_{k,\delta}^+(x) = \int_{|y-a_+| \leq C \log W / W^{1/2}} (A_*^+(x - a_+, y - a_+) + \tilde{A}_+(x, y)) \psi_{k,\delta}^+(y) dy + O(e^{-c \log^2 W}).$$

Here and below we denote

$$A_*^\pm := A_*^{(c_\pm)}, \quad A_m^\pm = P_m^\pm A P_m^\pm, \quad \tilde{A}_+(x, y) = A(x, y) - A_*^+(x - a_+, y - a_+),$$

where the projections P_m^+ and P_m^- are defined like in (5.3) for $\{\psi_{k,\delta}^+\}_{k=0}^m$ and $\{\psi_{k,\delta}^-\}_{k=0}^m$.

Expanding \mathcal{F}_0 for $|x - a_+| \leq CW^{-1/2} \log W$, $|y - a_+| \leq CW^{-1/2} \log W$, we get in this neighbourhood

$$\tilde{A}(x, y) = A(x, y) O(\log^3 W / W^{3/2}).$$

Thus, for $k \leq m$

$$A\psi_k^+(x) = (A_*^+ \psi_k)(x - a_+) + O(\log^3 W / W^{3/2}), \quad (5.4)$$

and similarly for A_1 of (3.2)

$$A_1\psi_k'(x) = (A_*^+ \psi_k)(x - b_s) + O(\log^3 W / W^{3/2}). \quad (5.5)$$

Remark 5.1 Applying the Taylor expansions up to the m -th order to the functions $\mathcal{F}_0(x)$ and $\mathcal{F}_0(y)$ one can prove that for $j, k = 0, \dots, m$

$$|A_{m,jk}^\pm| \leq \begin{cases} (Cm_{j,k}/W)^{|j-k|/2}, & j - k \geq 2; \\ (Cm_{j,k}/W)^{3/2}, & j - k = 1; \\ (Cm_{j,k}/W)^{|j-k|/2}, & k - j \geq 3; \\ (Cm_{j,k}/W)^{3/2}, & k - j = 1, 2. \end{cases}, \quad (5.6)$$

where $m_{j,k} = \max\{j, k\}$. In addition,

$$\begin{aligned} \|(A_\pm^{(12)})_j\| &\leq \begin{cases} (Cm/W)^{|m-j|/2}, & m - j \geq 2; \\ Cm/W, & m - j = 1, \end{cases} \\ \|(A_\pm^{(21)})_j\| &\leq \begin{cases} (Cm/W)^{|m-j|/2}, & m - j \geq 3; \\ (Cm/W)^{3/2}, & m - j = 1, 2, \end{cases} \end{aligned} \quad (5.7)$$

where $(A_\pm^{(12)})_j$ and $(A_\pm^{(21)})_j$ are the j -th row and column of $A_\pm^{(12)} = P_\pm A(I_\pm - P_\pm)$ and $A_\pm^{(21)} = (I_\pm - P_\pm)AP_\pm$ respectively (I_\pm here are operators of multiplication by $1_{\omega_\pm^\pm}$). Indeed, it is well known that the Hermite functions $\{\psi_k(x)\}_{k=0}^\infty$ satisfy the recursion relation

$$x\psi_k(x) = \sqrt{\frac{k+1}{4\alpha_1 W}} \psi_{k+1}(x) + \sqrt{\frac{k}{4\alpha_1 W}} \psi_{k-1}(x).$$

Hence, the operator \hat{L} of multiplication by $x - a_+$ has a three diagonal form in the basis $\{\psi_k^+\}$, and \hat{L}^l has $2l + 1$ non empty diagonals. The recursion relations combined with (5.2) yield (5.6). To prove (5.7) we have to use also

$$\begin{aligned} \|(A_\pm^{(12)})_j\|^2 &= \|A^* \psi_{j,\delta}^\pm\|^2 - \|(A_m^\pm)^* \psi_{j,\delta}^\pm\|^2 + O(e^{-cW}), \\ \|(A_\pm^{(21)})_j\|^2 &= \|A \psi_{j,\delta}^\pm\|^2 - \|A_m^\pm \psi_{j,\delta}^\pm\|^2 + O(e^{-cW}). \end{aligned}$$

Similar bounds hold for A_1 , thus for $K = A \otimes A_1$ (probably with multiplication by m^p with some absolute $p > 0$).

Proof of (4.14). By (4.9), to prove the bound for $\|\hat{K}^{(12)}\|$, we need to prove bounds for $\|P^+ \hat{K}(1 - P^+)\|$ and $\|P^- \hat{K}(1 - P^-)\|$. Let P_m and $P_{1,m}$ be the projections on $\{\psi_k(x - a_+)\}_{0 \leq k \leq m}$ and $\{\psi_k(x - b_s)\}_{0 \leq k \leq m}$. Then

$$\begin{aligned} \|P^+ \hat{K}(1 - P^+)\| &= \|(P_m \otimes P_{1,m})(\hat{A} \otimes \hat{A}_1)(1 - P_m \otimes P_{1,m})\| \\ &= \|(P_m \otimes P_{1,m})(\hat{A} \otimes \hat{A}_1)((1 - P_m) \otimes 1 + 1 \otimes (1 - P_{1,m}) - (1 - P_m) \otimes (1 - P_{1,m}))\| \\ &\leq \|\hat{A}^{(12)}\| + \|\hat{A}_1^{(12)}\| + \|\hat{A}^{(12)}\| \cdot \|\hat{A}_1^{(12)}\|. \end{aligned}$$

Hence it suffices to prove that

$$\|\hat{A}^{(12)}\| \leq Cm/W, \quad \|\hat{A}_1^{(12)}\| \leq Cm/W,$$

which follows from (5.7). By the same way one can estimate $\|(I^+ - P^+)KP^+\|$, $\|P^-K(I^- - P^-)\|$, $\|(I^- - P^-)KP^-\|$ and prove (4.16). \square

Proof of (4.13), (4.18), and (4.19). Since $\hat{K}^{(11)} = \hat{K}^+ \oplus \hat{K}^- + O(e^{-cW^2})$, it suffices to prove the bound for $\|(\hat{K}^+ - z)^{-1}\|$ and $\|(\hat{K}^- - z)^{-1}\|$. The bounds are very similar, hence we prove only the first one.

Consider the diagonal matrix with the entries

$$D_{\bar{k}\bar{k}} = A_{*k_1 k_1}^+ A_{*k_2 k_2}^+ - z, \quad 0 < |\bar{k}| \leq m.$$

According to (4.12), (4.11) and (5.1) we get for $|k| > 0$

$$\begin{aligned} |D_{\bar{k}\bar{k}}| &> \left| 1 - \frac{\alpha_+(2k_1 + 1)}{W} - \frac{\alpha_+(2k_2 + 1)}{W} + O(W^{-2}) - z \right| \\ &\geq \left| |z| - \left| 1 - \frac{\alpha_+(2k_1 + 1)}{W} - \frac{\alpha_+(2k_2 + 1)}{W} + O(W^{-2}) \right| \right| \\ &\geq \frac{\alpha_1(2k_1 + 2k_2 + 3/2)}{W} - \frac{5\alpha_1}{2W} + O(W^{-2}) = \frac{\alpha_1(2k_1 + 2k_2 - 1 - \varepsilon)}{W}. \end{aligned} \tag{5.8}$$

Hence

$$\|D^{-1}\| \leq CW.$$

Set $R = (\hat{K}^+ - D - z)D^{-1}$. Let Q be the matrix which contains $O(1)$ -order (or higher order) entries of R while the others entries are replaced by zeros. It follows from (5.2), (5.4) – (5.5) that

$$Q_{\bar{k}\bar{k}'} \neq 0 \quad \text{iff} \quad \bar{k}' - \bar{k} = 2e_1 \vee \bar{k}' - \bar{k} = 2e_2 \quad (e_1 = (1, 0), e_2 = (0, 1)). \tag{5.9}$$

Using the notations we can rewrite

$$\begin{aligned} (\hat{K}^+ - z)^{-1} &= D^{-1}(I + R)^{-1} = D^{-1}(1 + Q)^{-1}(I + \tilde{R})^{-1} \\ &= D^{-1}(1 + Q)^{-1}(I - \tilde{R}(I + \tilde{R})^{-1}), \end{aligned} \tag{5.10}$$

where $\tilde{R} = (R - Q)(I + Q)^{-1}$.

Moreover, there exists an absolute constant l_α such that for $|k| > l_\alpha$

$$\begin{aligned} |Q_{\bar{k}, \bar{k}+2e_1}| + |Q_{\bar{k}, \bar{k}+2e_2}| &\leq \frac{|A_{*k_1 k_1}^+ A_{*k_2, k_2+2}^+|}{\alpha_1(2k_1 + 2k_2 + 3 - \varepsilon)} + \frac{|A_{*k_1, k_1+2}^+ A_{*k_2, k_2}^+|}{\alpha_1(2k_1 + 2k_2 + 3 - \varepsilon)} \\ &\leq \frac{\alpha_2}{\alpha_1} \frac{\sqrt{(k_1 + 1)(k_1 + 2)} + \sqrt{(k_2 + 1)(k_2 + 2)}}{(k_1 + k_2 + (3 - \varepsilon)/2)} \leq (\alpha_2/\alpha_1)^{1/2} = q < 1. \end{aligned} \tag{5.11}$$

Here we used (5.1), (5.8) and the fact $\alpha_2 < \alpha_1$ (see (4.10) and use $\arg c_{\pm} \in (-\pi/2, \pi/2)$).

Write Q as a block matrix

$$\begin{aligned} Q^{(11)} &= \{Q_{\bar{k}, \bar{k}'}\}_{|k| \leq l_\alpha, |k'| \leq l_\alpha}, & Q^{(12)} &= \{Q_{\bar{k}, \bar{k}'}\}_{|k| \leq l_\alpha, |k'| > l_\alpha}, \\ Q^{(21)} &= \{Q_{\bar{k}, \bar{k}'}\}_{|k| > l_\alpha, |k'| \leq l_\alpha} & Q^{(22)} &= \{Q_{\bar{k}, \bar{k}'}\}_{|k| > l_\alpha, |k'| > l_\alpha} \end{aligned}$$

Then by (5.9) $Q^{(21)} = 0$, and by (5.11) $\|Q^{(22)}\| \leq q$. Moreover, (5.9) implies that for $s_0 = [l_\alpha/2] + 1$

$$Q^{s_0} = \begin{pmatrix} 0 & X \\ 0 & (Q^{(22)})^{s_0} \end{pmatrix} \Rightarrow Q^{s_0+p} = \begin{pmatrix} 0 & X(Q^{(22)})^p \\ 0 & (Q^{(22)})^{s_0+p} \end{pmatrix}, \quad p > 0,$$

where X is some fixed matrix.

Writing the Neumann series $(1 + Q)^{-1} = \sum_s (-1)^s Q^s$ we obtain that in view of (5.9)

$$|(1 + Q)_{\bar{k}, \bar{k}'}^{-1}| \leq Cq^{|\bar{k} - \bar{k}'|/2}, \quad (5.12)$$

and, in addition,

$$(1 + Q)_{\bar{k}, \bar{k}'}^{-1} = 0, \text{ if } \bar{k} - \bar{k}' \notin 2\mathbb{Z}^2. \quad (5.13)$$

Note that below $0 < q < 1$ can be different in different formulas.

Besides, it is easy to check using (5.6) and (5.8) that

$$\|R - Q\| \leq m^p W^{-3/2}, \quad |(R - Q)_{\bar{k}, \bar{k}'}| \leq (Cm/W)^{|\bar{k} - \bar{k}'|/2}.$$

Here and below we denote by p, p_1, p_2 etc. some absolute exponents which could be different in different formulas. Hence

$$|\tilde{R}_{\bar{k}, \bar{k}'}| = \left| \sum_{|\bar{k}''| \leq m} (R - Q)_{\bar{k}, \bar{k}''} (1 + Q)_{\bar{k}'', \bar{k}'}^{-1} \right| \leq C/W^{1/2} q^{|\bar{k} - \bar{k}'|/2}.$$

The last relation implies

$$|(1 + \tilde{R})_{\bar{k}, \bar{k}'}^{-1}| \leq Cq^{|\bar{k} - \bar{k}'|/2}. \quad (5.14)$$

To prove this, let us consider any fixed \bar{k} and \bar{k}' and use the standard trick from the spectral theory (see e.g. [15], Ch. 13.3). Assume that $|\bar{k} - \bar{k}'| = k_1 - k'_1$. Then denote D_q the diagonal matrix such that $(D_q)_{\bar{k}'', \bar{k}'''} = \delta_{\bar{k}'' \bar{k}'''} q^{k''_1/2}$. Then

$$\begin{aligned} \|D_q^{-1} \tilde{R} D_q\| &\leq C m^p / W^{1/2} \\ \Rightarrow |(1 + \tilde{R})_{\bar{k}, \bar{k}'}^{-1}| &= |(D_q(1 + D_q^{-1} \tilde{R} D_q)^{-1} D_q^{-1})_{\bar{k}, \bar{k}'}| \\ &\leq q^{(k_1 - k'_1)/2} \|(1 + D_q \tilde{R} D_q^{-1})^{-1}\|. \end{aligned}$$

If $|\bar{k} - \bar{k}'| = -(k_1 - k'_1)$ we use D_q^{-1} instead of D_q . And if $|\bar{k} - \bar{k}'| = \pm(k_2 - k'_2)$ we use $(D_q)_{\bar{k}'', \bar{k}'''} = \delta_{\bar{k}'' \bar{k}'''} q^{\pm k''_2/2}$. The last line of (5.10) combined with (5.14) proves (4.13) and the representation similar to (4.18) for $(\hat{K}^+ - z)^{-1}$ with

$$G^{(ev)} = W^{-1} D^{-1} (1 + Q)^{-1}, \quad G^{(r)} = -W^{-1/2} D^{-1} (1 + Q)^{-1} \tilde{R} (I + \tilde{R})^{-1}.$$

Conditions of the second line of (4.18) hold because of (5.8) and (5.12) – (5.14).

Now let us use the standard linear algebra formula

$$P^+ \hat{G} P^+ = (\hat{K}^+ - z - \tilde{R}_1)^{-1}, \quad \tilde{R}_1 = \hat{K}_+^{(12)} (K_+^{(22)} - z)^{-1} \hat{K}_+^{(21)},$$

where

$$K_+^{(12)} = P_+ K (I_+ - P_+), \quad K_+^{(22)} = (I_+ - P_+) K P_+, \quad K_+^{(21)} = (I_+ - P_+) K (I_+ - P_+).$$

Assume for the moment that (4.15) is known already, which gives (see (4.33))

$$\|(K^{(22)} - z)^{-1}\| \leq CW/m^{1/3}.$$

Together with (4.13) – (4.14) this implies

$$\|(K_+^{(22)} - z)^{-1}\| \leq mW.$$

Then we obtain for $|\bar{k}|, |\bar{k}'| \leq m$

$$\begin{aligned} |(\tilde{R}_1)_{\bar{k}, \bar{k}'}| &= |(\hat{K}_+^{(12)} (K_+^{(22)} - z)^{-1} \hat{K}_+^{(21)})_{\bar{k}, \bar{k}'}| = \left| \sum_{\bar{k}'', \bar{k}'''} \hat{K}_{+, \bar{k}, \bar{k}''}^{(12)} (K_+^{(22)} - z)^{-1}_{\bar{k}'', \bar{k}'''} \hat{K}_{+, \bar{k}'', \bar{k}'}^{(21)} \right| \\ &\leq CmW \sum_{\bar{k}'', \bar{k}'''} |\hat{K}_{+, \bar{k}, \bar{k}''}^{(12)}| \cdot |\hat{K}_{+, \bar{k}'', \bar{k}'}^{(21)}|, \end{aligned}$$

which together with (5.7) implies

$$|(\tilde{R}_1)_{\bar{k}, \bar{k}'}| \leq m^p (Cm/W)^{|\bar{k} - \bar{k}'|/2}.$$

Now, using the trick applied above to prove (5.14) and the formula

$$(\hat{K}^+ - z - \tilde{R}_1)^{-1} = (\hat{K}^+ - z)^{-1} - (\hat{K}^+ - z)^{-1} \tilde{R}_1 (I - \tilde{R}_1 (\hat{K}^+ - z)^{-1})^{-1}$$

one can obtain (4.18) from the representation for $(\hat{K}^+ - z)^{-1}$.

Representation (4.19) follows from (4.18), the definition of η , η^* (4.2), and (5.6).

□

Proof of (4.15). Let us split the integration domain \mathbb{R}^2 into three sub domains, according to the value of the functions \mathcal{F} , \mathcal{F}_1 . One of the possible splitting is

$$\begin{aligned} \Lambda_1 &= \{(a, b) : |\mathcal{F}(a)|, |\mathcal{F}_1(b)| \geq 1 - \delta/2\}, \\ \Lambda_3 &= \{(a, b) : 1 - \delta > |\mathcal{F}(a)|, |\mathcal{F}_1(b)|\}, \\ \Lambda_2 &= \mathbb{R}^2 \setminus (\Lambda_1 \cup \Lambda_3). \end{aligned}$$

Write

$$u = u_1 + u_2 + u_3,$$

where $u_i = u1_{(a,b) \in \Lambda_i}$. Since $\max_{(a,b) \in \Lambda_2 \cup \Lambda_3} |\mathcal{F}(a)\mathcal{F}_1(b)| = 1 - \delta/2$, we have

$$\begin{aligned} \|Ku\|^2 &\leq \|(\mathcal{F}\mathcal{F}_1)^2 u\|^2 \leq \|u_1\|^2 + (1 - \delta/2)^2 \|u_2 + u_3\|^2 \\ &= 1 - (1 - (1 - \delta/2)^2) \|u_2 + u_3\|^2 \\ &\Rightarrow \|u_2 + u_3\|^2 \leq C_0(1 - \|Ku\|^2). \end{aligned} \tag{5.15}$$

Here we used that the operator with the kernel $B(a_1, a_2)$ defined by (3.2) satisfies the bound $\|B\| \leq 1$. This is true, if the integration with respect to a_1, a_2 is over the real line. If the integration contour is deformed (see Remark 3.1), then

$$\|B\| \leq \sup_a |\cos^{-1/2} 2\phi(a)|.$$

But if the condition (3.9) is satisfied, then the inequality (5.15) is still true (may be with some different C_0).

Moreover,

$$\begin{aligned} \Re(K(u_1 + u_2), Ku_3) &= \Re(Ku_1, Ku_3) + \Re(Ku_2, Ku_3) \\ &= O(e^{-cW^2}) + \Re(Ku_2, Ku_3) \leq O(e^{-cW^2}) + \frac{1}{2}\|u_2 + u_3\|^2, \end{aligned} \quad (5.16)$$

since $\|K\| \leq 1$. Denote

$$u_0 := u_1 + u_2, \quad u_0^+ = u_0 1_{\omega_\delta^+}, \quad u_0^- = u_0 1_{\omega_\delta^-}. \quad (5.17)$$

Lemma 5.2 *For u_0^+ and u_0^- defined in (5.17) we have*

$$\|Ku_0^+\|^2 \leq (1 - Cm^{1/3}/W)\|u_0^+\|^2, \quad \|Ku_0^-\|^2 \leq (1 - Cm^{1/3}/W)\|u_0^-\|^2. \quad (5.18)$$

Assume for the moment that the lemma is proved and finish the proof of (4.15).

We have by (5.15), (5.16), and the lemma

$$\begin{aligned} \|Ku\|^2 &= \|K(u_0^+ + u_0^- + u_3)\|^2 \\ &= \|Ku_0^+\|^2 + \|Ku_0^-\|^2 + 2\Re(Ku_0, Ku_3) + \|Ku_3\|^2 + O(e^{-cW}) \\ &\leq (1 - Cm^{1/3}/W)(\|u_0^+\|^2 + \|u_0^-\|^2) + 2\|u_2 + u_3\|^2 + O(e^{-cW}) \\ &\leq 1 - Cm^{1/3}/W + 2C_0(1 - \|Ku\|^2) \\ &\Rightarrow (1 - \|Ku\|^2)(1 + 2C_0) \geq Cm^{1/3}/W \\ &\Rightarrow \|Ku\|^2 \leq 1 - C_1 m^{1/3}/W. \end{aligned}$$

Here we used that

$$\|Ku_3\|^2 \leq \|u_3\|^2 \leq \|u_2\|^2 + \|u_3\|^2 = \|u_2 + u_3\|^2.$$

□

Proof of Lemma 5.2.

Choose $c_0 > 0$ sufficiently small to provide

$$\Re f(x) \geq \frac{c_0}{2}(x - a_+)^2, \quad x > 0,$$

and denote

$$\mathcal{F}_0 = e^{-c_0(x-a_+)^2/2}, \quad A_0 = \mathcal{F}_0 B \mathcal{F}_0.$$

Consider the basis $\{\tilde{\psi}_k\}_{k \geq 0}$ defined by (4.5) for c_0 . Define the operator kernel $A_{0,+}(a_1, a_2) := A_0(a_1, a_2)$ and similarly define $A_{0,1}(b_1, b_2)$ (with b_s instead of a_+). Since c_0 is real, $A_{0,+}$ and $A_{0,1}$ are diagonal in the basis $\{\tilde{\psi}_k\}_{k \geq 0}$. Moreover, the commutator $[\mathcal{F}, B]$ admits the bound

$$\|[\mathcal{F}, B]\| \leq \sup_x W \int |\mathcal{F}(x) - \mathcal{F}(y)| e^{-W^2(x-y)^2} dy \leq CW \int |x-y| e^{-W^2(x-y)^2} dy \leq C_*/W,$$

and by the same argument

$$\|[\mathcal{F}_0, B]\| \leq C_*/W.$$

Thus, denoting Λ^+ the projection on ω_δ^+ we get

$$\begin{aligned} \Lambda^+ A^* \Lambda^+ A \Lambda^+ &= \Lambda^+ \mathcal{F}^* B \mathcal{F}^* \Lambda^+ \mathcal{F} B \mathcal{F} \Lambda^+ = \Lambda^+ B \mathcal{F}^* \mathcal{F}^* \Lambda^+ \mathcal{F} B \mathcal{F} \Lambda^+ + \Lambda^+ [\mathcal{F}^*, B] \mathcal{F}^* \Lambda^+ \mathcal{F} B \mathcal{F} \Lambda^+ \\ &\quad + \Lambda^+ B \mathcal{F}^* \mathcal{F}^* \Lambda^+ \mathcal{F} [B, \mathcal{F}] \Lambda^+ + \Lambda^+ [\mathcal{F}^*, B] \mathcal{F}^* \Lambda^+ \mathcal{F} [B, \mathcal{F}] \Lambda^+ \\ &\leq \Lambda^+ B \mathcal{F}^* \mathcal{F}^* \Lambda^+ \mathcal{F} B \mathcal{F} \Lambda^+ + 3C_* W^{-1} \\ &\leq \Lambda^+ B \mathcal{F}_0^4 B \Lambda^+ + 3C_* W^{-1} \leq \Lambda^+ \mathcal{F}_0 B \mathcal{F}_0^2 B \mathcal{F}_0 \Lambda^+ + 6C_* W^{-1} \\ &= \Lambda^+ A_0 A_0 \Lambda^+ + 6C_* W^{-1}, \end{aligned}$$

and similarly

$$\Lambda_1 A_1^* \Lambda_1 A_1 \Lambda_1 \leq \Lambda_1 A_0^2 \Lambda_1 + 6C_* W^{-1},$$

where Λ_1 is the projection on $\omega_{1,\delta}$. Set

$$K_0 = A_{0,+} \otimes A_{0,1}, \quad \Lambda = \Lambda^+ \otimes \Lambda_1.$$

Then, taking into account that $\|A\|, \|A_1\| \leq 1$, we obtain

$$\begin{aligned} \Lambda K^* \Lambda K \Lambda &= \Lambda^+ A^* \Lambda^+ A \Lambda^+ \otimes \Lambda_1 A_1^* \Lambda_1 A_1 \Lambda_1 \leq \Lambda^+ A_0^2 \Lambda^+ \otimes \Lambda_1 A_1^* \Lambda_1 A_1 \Lambda_1 + 6C_* W^{-1} \quad (5.19) \\ &\leq \Lambda^+ A_0^2 \Lambda^+ \otimes \Lambda_1 A_{0,1}^2 \Lambda_1 + 12C_* W^{-1} = \Lambda K_0^2 \Lambda + 12C_* W^{-1}. \end{aligned}$$

Let \tilde{P}_l and $\tilde{P}_{1,l}$ be the projection operator on $\{\tilde{\psi}_k(x - a_+)\}_{0 \leq k \leq l}$ and $\{\tilde{\psi}_k(x - b_s)\}_{0 \leq k \leq l}$ respectively, while P_m and $P_{1,m}$ be the projections on $\{\psi_k(x - a_+)\}_{0 \leq k \leq m}$ and $\{\psi_k(x - b_s)\}_{0 \leq k \leq m}$. By (5.3)

$$\begin{aligned} &\|\tilde{P}_l \otimes \tilde{P}_{1,l} (1 - P_m \otimes P_{1,m})\| \quad (5.20) \\ &= \|\tilde{P}_l \otimes \tilde{P}_{1,l} ((1 - P_m) \otimes I + I \otimes (1 - P_{1,m}) - (1 - P_m) \otimes (1 - P_{1,m}))\| \\ &\leq \|\tilde{P}_l(1 - P_m)\| + \|\tilde{P}_{1,l}(1 - P_{1,m})\| + \|\tilde{P}_l(1 - P_m)\| \cdot \|\tilde{P}_{1,l}(1 - P_{1,m})\| \leq \frac{Cl^3}{m} \leq \frac{1}{2}, \end{aligned}$$

if $l = m^{1/3}/C_1$ with sufficiently big C_1 . Use now the following proposition

Proposition 5.1 *Let $\mathcal{A} \leq I$ and $\mathcal{A}_0 \leq I$ be positive operators, Λ , P and \tilde{P} be projection operators such that*

$$\begin{aligned} \Lambda \mathcal{A} \Lambda &\leq \Lambda \mathcal{A}_0 \Lambda + \delta, \quad \|[\Lambda, P]\| \leq \delta_1, \quad \|[\Lambda, \tilde{P}]\| \leq \delta_1, \\ [\tilde{P}, \mathcal{A}_0] &= 0, \quad (1 - \tilde{P}) \mathcal{A}_0 (1 - \tilde{P}) \leq 1 - \Delta, \\ \|\tilde{P}(1 - P)\| &\leq \frac{1}{2}. \end{aligned} \quad (5.21)$$

Then

$$\Lambda(1 - P) \mathcal{A} (1 - P) \Lambda \leq 1 - \Delta/2 + \delta + 4\delta_1. \quad (5.22)$$

Apply the proposition to $\mathcal{A} = K^* \Lambda K$, $\mathcal{A}_0 = K_0$, $P = P^+ = P_m \otimes P_{1,m}$, $\tilde{P} = \tilde{P}_l \otimes \tilde{P}_{1,l}$. Then $\Delta = l\sqrt{2c_0}/2W$ since \tilde{P} is a correspondent spectral projection of K_0 , $\delta = 8C_* W^{-1}$ (by (5.19), $\delta_1 = O(e^{-cW})$), and (5.21) is valid in view (5.20). Then (5.22) yields (5.18).

□

Proof of Proposition 5.1.

Take any $u = (1 - P)\Lambda v$, $\|v\| = 1$. Then

$$\begin{aligned} (\mathcal{A}u, u) &\leq ((1 - P)\Lambda\mathcal{A}\Lambda(1 - P)v, v) + 2\delta_1 \leq (\mathcal{A}_0u, u) + \delta + 4\delta_1 \\ &= (\tilde{P}\mathcal{A}_0\tilde{P}u, u) + ((1 - \tilde{P})\mathcal{A}_0(1 - \tilde{P})u, u) + \delta + 4\delta_1 \\ &\leq \|\tilde{P}u\|^2 + (1 - \Delta)(\|u\|^2 - \|\tilde{P}u\|^2) + \delta + 4\delta_1 \\ &\leq (1 - \Delta) + \Delta\|\tilde{P}u\|^2 + \delta + 2(2\delta_1 + \delta_1^2) \leq (1 - \Delta/2) + \delta + 4\delta_1, \end{aligned}$$

since

$$\begin{aligned} |((1 - \tilde{P})\mathcal{A}_0(1 - \tilde{P})u, u)| &= |((1 - \tilde{P})^2\mathcal{A}_0(1 - \tilde{P})^2u, u)| \leq \|(1 - \tilde{P})u\|^2 = \|u\|^2 - \|\tilde{P}u\|^2; \\ \|\tilde{P}u\|^2 &= \|\tilde{P}(1 - P)\Lambda v\|^2 \leq \|\Lambda v\|^2/2 \leq 1/2. \end{aligned}$$

□

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References

- [1] Bao, J., Erdős, L.: Delocalization for a class of random block band matrices, arXiv:1503.07510
- [2] Bogachev, L. V., Molchanov, S. A., and Pastur, L. A.: On the level density of random band matrices. Mat. Zametki, **50:6**, 31 – 42 (1991)
- [3] Bourgade, P., Erdős, L., Yau, H.-T., Yin, J. Universality for a class of random band matrices, arXiv:1602.02312
- [4] Campanino, M., Klein, A.: A supersymmetric transfer matrix and differentiability of the density of states in the one-dimensional Anderson model. Comm. Math. Phys. **104**, p. 227 – 241 (1986)
- [5] Casati, G., Molinari, L., Israilev, F.: Scaling properties of band random matrices, Phys. Rev. Lett. **64** (1990), 1851–1854.
- [6] Constantinescu, F.: The supersymmetric transfer matrix for linear chains with nondiagonal disorder. J. Stat. Phys. **50**, p. 1167 – 1177 (1988)
- [7] Disertori, M., Pinson, H., and Spencer, T.: Density of states for random band matrices. Comm. Math. Phys. **232**, 83 – 124 (2002)
- [8] Disertori, M., Sodin, S.: Semi-classical analysis of non self-adjoint transfer matrices in statistical mechanics. I, Annal. Henri Poincaré, <http://dx.doi.org/10.1007/s00023-015-0397-x> (2015)

- [9] Efetov, K.: Supersymmetry in disorder and chaos. Cambridge university press, New York (1997)
- [10] Erdős, L., Knowles, A.: Quantum diffusion and eigenfunction delocalization in a random band matrix model. *Commun. Math. Phys.* **303**, 509 – 554 (2011).
- [11] Erdős, L., Knowles, A., Yau, H.-T., Yin, J.: Delocalization and diffusion profile for random band matrices, *Commun.Math.Phys.* **323**, 367 – 416 (2013).
- [12] Erdős, L., Yau, H.-T., Yin, J.: Bulk universality for generalized Wigner matrices, *Probab. Theory Relat. Fields* **154**, 341 – 407 (2012)
- [13] Fyodorov, Y.V., Mirlin, A.D.: Scaling properties of localization in random band matrices: a σ -model approach, *Phys. Rev. Lett.* **67**, 2405 – 2409 (1991).
- [14] Molchanov, S. A., Pastur, L. A., Khorunzhi, A. M.: Distribution of the eigenvalues of random band matrices in the limit of their infinite order, *Theor. Math. Phys.* **90**, 108 – 118 (1992)
- [15] Pastur, L. A., Shcherbina, M. Eigenvalue distribution of large random matrices. American mathematical society, 2011.
- [16] Schenker, J.: Eigenvector localization for random band matrices with power law band width, *Commun. Math. Phys.* **290**, 1065 – 1097 (2009)
- [17] Shcherbina, M., Shcherbina, T. Characteristic polynomials for 1D random band matrices from the localization side, arXiv:1602.08737
- [18] Shcherbina, T. : On the second mixed moment of the characteristic polynomials of the 1D band matrices. *Commun. Math. Phys.* **328**, p. 45 – 82 (2014)
- [19] Shcherbina, T.: Universality of the local regime for the block band matrices with a finite number of blocks. *J.Stat.Phys.* **155**, 3, p. 466 – 499 (2014)
- [20] Sodin, S.: An estimate for the average spectral measure of random band matrices. *J. Stat. Phys.* **144**, p. 46 – 59 (2011)
- [21] Spencer, T.: SUSY statistical mechanics and random band matrices. Quantum many body system, Cetraro, Italy 2010, Lecture notes in mathematics 2051 (CIME Foundation subseries) (2012)